On the Congruency-Constrained Matroid Base ⋆

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Abstract. Consider a matroid where all elements are labeled with an element from $\mathbb{Z}_m$. We are interested in finding a base where the sum of the labels is congruent to $g \pmod m$. We show that this problem can be solved in $O(m^{2m} nr \log r)$ time for a matroid with n elements and rank r, when m is either the product of two primes or a prime power. The algorithm generalizes to all moduli, and in fact, to all abelian groups, if a classic additive combinatorics conjecture of Schrijver and Seymour is true. We also discuss the optimization version of the problem.

Keywords: Matroid · Additive Combinatorics · Optimization.

1 Introduction

Recently, there has been a surge of work on congruency-constrained combinatorial optimization problems, such as submodular minimization [19], constraint satisfaction problems [5], and integer programming over totally unimodular matrices [1,18]. In this paper, we consider congruency-constrained matroid base problems.

As an alternative motivation, consider the classical subset sum problem: given a set of natural numbers $x_1, \ldots, x_n$ and a target number t, does there exist a subset whose sum equals t? It is known that the problem, along with the optimization variant, the knapsack problem, can be solved in pseudopolynomial time [4].

Additionally, one can impose further restrictions on the subsets chosen in the subset sum problem, for example, by adding a cardinality constraint. The cardinality-constrained subset sum problem can also be solved in pseudopolynomial time. A natural progression of constraints leads to matroid base constraints, which correspond to the following problem.

Problem 1 (Exact matroid base). For a matroid $M = (E, \mathcal{I})$ and a natural number t. Let there be a function $\ell : E \to \mathbb{N}$. Find a base B in M such that $\sum_{x \in B} \ell(x) = t$, if any.

⋆ We became aware after our submission that this problem was posed at 14th Emléktábla Workshop.

Papadimitriou and Yannakakis raised the question of whether the above problem can be solved in pseudopolynomial time [21]. Barahona and Pulleyblank provided an affirmative answer for graphic matroids [3]. Subsequently, Camerini and Maffioli demonstrated that for matroids representable over the reals, there is a randomized pseudopolynomial time algorithm for the exact matroid base problem [10]. However, it remains unclear whether this algorithm can be derandomized or whether an algorithm exists for non-linear matroids. For one possible optimization variant, where the goal is to maximize the value of a base while adhering to an upper bound on the budget, some approximation results are known [9,8].

Instead of focusing on an exact integer solution, we can alternatively consider a modular version. Specifically, we study the following problem.

Problem 2 (m-Congruency-Constrained Matroid Base (CCMB(m))). Let matroid $M = (E, \mathcal{I})$, $g \in \{0, \dots, m-1\}$ and a label function $\ell : E \to \mathbb{Z}$. Find a base B in M such that $\sum_{x \in B} \ell(x) \equiv g \pmod{m}$, if any.

As far as we know, except results implied by exact matroid base, little is known for this problem. A folklore algorithm with running time polynomial in m and size of the matroid imply existence of pseudopolynomial time algorithm for exact matroid base. We ask for the next best thing: does there exist a fixed-parameter tractable (FPT) algorithm for CCMB(m) parameterized by m?

The closest study related to matroid bases and congruency constraints comes from the additive combinatorics community, where theorems considering the number of different labels attainable by the bases were discovered [22,13].

Our Contribution We demonstrate that for each m, being either a prime power or a product of two primes, CCMB(m) can be solved by an FPT algorithm parameterized by m. Furthermore, we establish that an FPT algorithm would exist for all abelian groups when parameterized by the group size, assuming that a certain additive combinatorics conjecture by Schrijver and Seymour is valid. These results are obtained through a proximity result that relates to the congruency-constrained bases. Preliminary results on the optimization version of CCMB are also presented.

Organization In **??**, we define the problem and provide the necessary background in matroid and group theory. In Section 3, we discuss established algorithms for congruency-constrained bases and demonstrate the existence of an FPT algorithm contingent upon a proximity result. Section 4 explains the process of reducing the problem across all matroids to block matroids of fixed rank. Finally, Section 5 and Section 6 address the congruency-constrained base and its optimization variant, respectively.

2 Preliminaries

Similar to [17], we will consider finite abelian groups instead of solely congruency constraints, as this approach simplifies the presentation. Let G be a finite (additive) abelian group. Let $M = (E, \mathcal{I})$ be a matroid, where E is the ground set and $\mathcal{I}$ is the family of independent sets. Let $\ell : E \to G$ be a label function mapping each element of the ground set to an element of G. For any $F \subseteq E$, denote $\ell(F) := \sum_{e \in F} \ell(e)$. ℓ is called a *G-labeling* of E. For a base B, we will say B is a *g-base* if $\ell(B) = g$.

Problem 3 (Group-constrained matroid base (GCMB(G))). Given $g \in G$, a matroid $M = (E, \mathcal{I})$, and a label function $\ell : E \to G$, find a g-base, if one exists.

Problem 4 (Group-constrained optimum matroid base (GCOMB(G))). Given $g \in G$, a matroid $M = (E, \mathcal{I})$, a label function $\ell : E \to G$, and a weight function $w : E \to \mathbb{R}$, find a g-base, if one exists, with the minimum weight.

Certainly, solving GCOMB in polynomial time implies the solvability of GCMB in polynomial time. We assume that the matroid can be accessed through an independence oracle that takes a constant time per query.

Notation. Given a matroid $M = (E, \mathcal{I})$, a group G, and $\ell : E \to G$. For each $g \in G$, denote all elements with label g by $E(g) := \ell^{-1}(g) = \{e \in E \mid \ell(e) = g\}$. For $H \subseteq G$, denote $E(H) := \ell^{-1}(H) = \{e \in E \mid \ell(e) \in H\}$. Let $\mathcal{B}$ be the set of bases of M. Denote the collection of all g-bases of M by $\mathcal{B}(g) := \{B \in \mathcal{B} \mid \ell(B) = g\}$. Denote by $\ell(M)$ the set $\{\ell(B) \mid B \text{ is a base of } M\}$. For $F \subseteq E$, denote the rank of F in M by $r_M(F)$, where M in the rank function $r_M(\cdot)$ can be omitted if the context is clear. Denote the rank of matroid M by $r(M) := r_M(E)$.

Given a matroid $M = (E, \mathcal{I})$ and a subset $F \subseteq E$, let $M \setminus F$ denote the *deletion* minor of M whose independent sets are those of M, *restricted* to $E \setminus F$. If $F \subseteq E$ is an independent set of M, let M/F denote the *contraction* minor of M. A subset $A \subseteq E \setminus F$ is independent in M/F if and only if $A \cup F$ is independent in M. Their rank functions satisfy the relation $r_{M/F}(A) = r_M(A \cup F) - r_M(F)$. We assume the matroid is loopless throughout, which means $\{e\} \in \mathcal{I}$ for any $e \in E$.

A matroid possesses the base exchange property: For any two bases A and B, there exists a series of elements $a_1, \dots, a_k \in A \setminus B$ and $b_1, \dots, b_k \in B \setminus A$ such that $A - (\bigcup_{i=1}^{j} a_i) + (\bigcup_{i=1}^{j} b_i)$ is also a base for each $j = 1, \dots, k$, and $A - (\bigcup_{i=1}^{k} a_i) + (\bigcup_{i=1}^{k} b_i) = B$. The distance $d(A, B)$ between two bases A and B is the number of steps needed to exchange elements from A to B, which equals $|A \setminus B|$. The distance is symmetric since all bases have the same cardinality. The distance between a set A and a family of sets $\mathcal{B}$ is defined by $d(A, \mathcal{B}) = \min_{B \in \mathcal{B}} d(A, B)$. The asymmetric Hausdorff distance between two families of sets $\mathcal{A}$ and $\mathcal{B}$ is defined by $d_h(\mathcal{A}, \mathcal{B}) = \max_{A \in \mathcal{A}} d(A, \mathcal{B})$. A labeling ℓ of matroid M with bases $\mathcal{B}$ is called *k-close* if $d_h(\mathcal{B}, \mathcal{B}(g)) \le k$ for all $g \in G$ such that $\mathcal{B}(g) \ne \emptyset$. That is, if g-bases exist, then every base has a g-base differing by at most k elements. If every G-labeling is k-close for every matroid, then we say G is k-close.

For a weight function $w : E \to \mathbb{R}$, denote by $\mathcal{B}^* := \arg\min_{B \in \mathcal{B}} w(B)$ and $\mathcal{B}^*(g) := \arg\min_{B \in \mathcal{B}(g)} w(B)$ the sets of optimum bases and optimum g-bases, respectively. A labeling ℓ of matroid M is *strongly-k-close* if, for every weight w and $g \in G$ such that $\mathcal{B}(g) \neq \emptyset$, we have $d_h(\mathcal{B}^*, \mathcal{B}^*(g)) \leq k$. That is, if g-bases exist, then every *optimum base* has an *optimum g-base* differing by at most k elements. If every G-labeling is strongly k-close for

every matroid, then G is said to be strongly-k-close. Note that being strongly-k-close implies k-close by setting $w \equiv 0$.

A matroid is *strongly base orderable*, if for any two bases A and B, there is a bijection $f : A \to B$ such that for any $X \subseteq A$, we have $A - X + f(X)$ is a base. The function with the previous property can be taken to satisfy $f(a) = a$ for all $a \in A \cap B$. A matroid is a *block matroid* if the goundset is the union of two disjoint bases, and such disjoint bases are called *blocks*.

Given a group G and its normal subgroup $H \subseteq G$, denote by G/H the quotient group consisting of equivalent classes of the form $gH := \{g+h \mid h \in H\}$, which are called the *cosets* of H, where g_1 is equivalent to g_2 if and only if $g_1^{-1} g_2 \in H$. For a subset $F \subseteq G$, the *stabilizer* of F is defined by $\text{stab}(F) := \{g \in G \mid g + F = F\}$. It is easy to see $\text{stab}(F)$ is a subgroup of G.

The *Davenport constant* [12,20] of G, denoted by $D(G)$, is the minimum value such that every sequence of elements from G of length $D(G)$ contains a non-empty subsequence that sums to 0. In other words, the longest sequence without non-empty subsequence that sums to 0 has length $D(G) - 1$. By the Fundamental Theorem of Finite Abelian Groups, any finite abelian group G can be decomposed into $G = \mathbb{Z}_{m_1} \times \ldots \times \mathbb{Z}_{m_r}$ where $1 \mid m_1 \mid m_2 \mid ... \mid m_r$. Here $\mathbb{Z}_m$ is the group of integers modulo m, and $\times$ is the group direct product. Define $M(G) := \sum_i^r (m_i - 1) + 1$. A trivial example shows that $D(G) \geq M(G)$. It was proved independently by Olson [20] and Kruswijk [2] that $D(G) = M(G)$ for p-groups and for $r = 1, 2$. It also trivially holds that $D(G) \leq |G|$, where equality holds for all cyclic groups [14].

Our paper makes progress in the following two conjectures.

Conjecture 1 (Feasibility). Any finite abelian group G is $(D(G) - 1)$-close.

Conjecture 2 (Optimization). Any finite abelian group G is strongly-$(D(G)-1)$-close.

Note these conjectures are best possible for any group. Indeed, just take a block matroid of rank $D(G) - 1$, where A and B are disjoint blocks. By the definition of $D(G)$, there exists a labeling of the elements in A such that no subset sums to 0. Let all elements in B be labeled with 0. Then, B is the closest 0-base to A, which means $d(A, \mathcal{B}(0)) = D(G)-1$. Therefore, G is not $(D(G)-2)$-close.

3 Algorithms for GCOMB

In this section we describe the applicability k-close and strongly-k-close properties to FPT algorithms.

3.1 Algorithms through strongly-k-closeness

If G is strongly-k-close, then it certainly gives a polynomial time algorithm: find an optimum base B, consider all possible bases B' such that $|B \setminus B'| \leq k$, and return the optimum base B' with the desired label. The running time would be $O(r^k n^k)$ by enumerating all bases k-close to the optimum base.

As an example, we show that $\mathbb{Z}_2$ is strongly-1-close. First, we need an lemma.

Lemma 1 ([6]). *Let A and B be bases in a matroid, then there exists a bijection $f : A \setminus B \to B \setminus A$, such that $A - a + f(a)$ is a base.*

Proposition 1. *$\mathbb{Z}_2$ is strongly-1-close.*

Proof. Let A be an optimum base, and without loss of generality, let B be an optimum 0-base, and f be a bijection in Lemma 1. If A is a 0-base, then we are done. Otherwise, A is a 1-base. Note $w(A-a+f(a)) = w(A)-w(a)+w(f(a)) \geq w(A)$ for each $a \in A \setminus B$, so $-w(a) + w(f(a)) \geq 0$ for each $a \in A \setminus B$. There exists $a \in A \setminus B$ such that $A - a + f(a)$ is a 0-base. Indeed, otherwise A and B must have the same parity. Note $w(A-a+f(a)) = w(A)+(-w(a)+w(f(a))) \leq w(A) + \sum_{a \in A \setminus B}(-w(a) + w(f(a))) = w(B)$. Hence $A - a + f(a)$ is a optimum 0-base, which has distance 1 from A.

The algorithm specializes to $\mathbb{Z}_2$ is extremely simple: find a optimum base B, if it has the correct parity, then we are done. Otherwise, find an element $e \in B$ and $f \notin B$ such that $B - e + f$ is a base, $\ell(e) \neq \ell(f)$, and under such constraint $\ell(e) - \ell(f)$ is minimized. Return $B - e + f$ as an optimum base with the correct parity.

3.2 Algorithms through matroid intersection

There is a folklore algorithm which shows for any fixed group G, GCOMB(G) is solvable in polynomial time by reducing it to a polynomial number of matroid intersections.

Let $M = (E, \mathcal{I})$ be a matroid of size n and rank r. Recall $E(g) = \ell^{-1}(g)$ for $g \in G$. For a base B of matroid M, a vector $a \in \mathbb{N}^G$ such that $a_g = |B \cap E(g)|$ is called the *signature vector* of B. Every element $h \in G$ can be expressed as $\sum_{g \in G} a_g \cdot g = h$ for some $a \in \mathbb{N}^G$. Here $a \cdot g$ is defined to be $\sum_{i=1}^{a} g$. For a fixed vector a, if $\sum_g a_g = h$, $0 \leq a_g \leq |E(g)|$ and $\sum_g a_g \cdot g = h$, we can check if any base has signature a, in fact, a minimum cost base with signature a. Indeed, one can obtain this through matroid intersection with a partition matroid, whose partition classes are $\{E(g) \mid g \in G\}$. It suffices to find an optimum base of matroid M such that its intersection with $E(g)$ has cardinality a_g for each $g \in G$. So the number of vectors we need to check is bounded above by $\binom{r+|G|-1}{|G|-1} \leq r^{|G|-1}$. Hence, we have to run $r^{|G|-1}$ times matroid intersection in order to find the optimum g-base. Currently, the fastest algorithm for matroid intersection has running time $O(nr \log r)$, assuming independence oracle takes constant time [11]. Hence the total running time is $\tilde{O}(nr^{|G|})$ which is polynomial as $|G|$ is fixed.

Although the algorithm is polynomial time for fixed G, if we parameterize by the size of G, then this algorithm is not FPT.

3.3 Combine matroid intersection and k-closeness

There is a close relation between k-close property and the existence of an FPT algorithm.

Theorem 1. *If G is k-close or strongly-k-close, then there is a $\tilde{O}(|G|^{2k}nr)$ running time algorithm for GCMB(G) or GCOMB(G), respectively.*

Proof. The algorithm consists of two parts. First find any base B. By the property of (strongly)-k-close, we know there is a feasible (optimum) g-base D that differs from B in at most k positions. Let a_D and a_B be the signature vectors of D and B, respectively. Let $a^+ := (a_D - a_B) \vee 0$, $a^- := (a_D - a_B) \wedge 0$ be vectors taking coordinate-wise maximum and minimum of $(a_D - a_B)$ with 0, respectively. Note that $a^+ \geq 0$ and $\sum_g a_g^+ \leq k$. Therefore, there are at most $\binom{k+|G|-1}{k} \leq |G|^k$ possible a^+'s, and similarly at most $|G|^k$ possible a^-'s. Hence, we only need to run at most $|G|^{2k}$ times matroid intersection to search for the (optimum) g-bases which are (strongly)-k-close to B.

In particular, this shows if there is a function f such that G is $f(G)$-close or strongly-$f(G)$-close for each G, then GCMB(G) or GCOMB(G) is in FPT, respectively.

4 Property of minimum counterexamples

If G is not strongly-k-close, then there exists a counterexample of minimum size. Here we bound the size of such matroid.

We start with a lemma.

Lemma 2. *Given a matroid $M = (E, \mathcal{I})$ and any weight function $w : E \to \mathbb{R}$, let A be an optimum base such that its weight is minimized. Then, for any $a \in A$, there exists some $b \in E \setminus A$ such that $A - a + b$ is an optimum base in $M' = M \setminus a$ with weight being the restriction of w to $E \setminus a$.*

Proof. Suppose A' is an optimum base in $M \setminus a$. By [7], there exists a bijection $f : A \setminus A' \to A' \setminus A$ such that $A - e + f(e)$ is a base for any $e \in A \setminus A'$. Since A is an optimum base, we have $w(A) \leq w(A - e + f(e))$, and thus $w(e) \leq w(f(e))$ for each $e \in A \setminus A'$. Let $b = f(a)$. Then, $w(A - a + b) = w(A) - w(a) + w(b) \leq w(A) - \sum_{e \in A \setminus A'}(w(e) - w(f(e))) = w(A')$. The inequality is because $a \in A \setminus A'$ and $w(e) \leq w(f(e))$ for any $e \in A \setminus A' \setminus a$. Since A' is an optimum base in $M \setminus a$, $A - a + b$ is also an optimum base in $M \setminus a$.

Next, we show that if G is not strongly-k-close, then there has to be a G-labeling ℓ, a weight function w, $g \in G$, and a block matroid whose blocks are A, B, such that A is an optimum base, B is the closest optimum g-base from A, and $|A \setminus B| > k$. We call the bases A and B a *witness.*

Theorem 2. *If G is not strongly-k-close, then the there is a block matroid M of rank $k + 1$ with blocks A and B that forms a witness.*

Proof. Let matroid $M = (E, \mathcal{I})$ be the smallest matroid, in terms of the number of elements, such that G is not strongly-k-close for M. Let $w : E \to \mathbb{Z}$ be a weight function and $g \in G$. Recall that $\mathcal{B}^* := \arg\min_{B\in\mathcal{B}} w(B)$ and $\mathcal{B}^*(g) := \arg\min_{B\in\mathcal{B}(g)} w(B)$ are the sets of optimum bases and optimum g-bases, respectively. There exists $A \in \mathcal{B}^*$ and $B \in \mathcal{B}^*(g)$, such that B is the optimum g-base closest to A but $|A \setminus B| > k$.

First, we argue that $A \cap B = \emptyset$. If not, let $M' = (E', \mathcal{I}')$ be the matroid obtained by contracting $A \cap B$, and let $A' = A \setminus B$, $B' = B \setminus A$, $g' = g - \ell(A \cap B)$ and $w' = w|_{E'}$. Clearly, A' is a weighted minimum base of M' and B' is a weighted minimum g'-base of M' with weight w'. Also, $|A' \setminus B'| = |A \setminus B| > k$, and B' is the optimum g'-base closest to A'. However, $|E'| = |E| - |A \cap B| < |E|$, contradicting to the minimality of $|E|$. Moreover, $A \cup B = E$. Otherwise, let M' be the matroid obtained by deleting $E \setminus (A \cup B)$ and A, B stays a witness in M', contradicting to the minimality of $|E|$. Therefore, M is a block matroid which is the disjoint union of bases A, B.

Suppose $r(M) > k + 1$. Pick any $a \in A$. Let $M' = (E', \mathcal{I}) := M \setminus a$. By Lemma 2, there exists $b \in B$, such that $A' := A - a + b$ is a weigted minimum base in M'. Clearly, B is still a weighted minimum g-base in M' that is closest to A'. Moreover, $|A' \setminus B| = |A \setminus B| - 1 = |A| - 1 = r(M) - 1 > k$. Therefore, G is not strongly-k-close for M', since A', B forms a witness. Because $|E'| < |E|$, we obtain a contradiction. Therefore, $r(M) \le k + 1$. Combining the fact that $r(M) = |A \setminus B| > k$, we get $r(M) = k + 1$.

As a corollary, if G is not k-close, then there has to be a G-labeling ℓ, $g \in G$ and a block matroid M with blocks A, B, such that A is a base, B is the closest g-base from A, and $|A \setminus B| > k$. Similarly, we call the bases A and B a *witness* in the unweighted setting.

Corollary 1. *If G is not k-close, then the there is a block matroid M of rank $k+1$ with blocks A and B that forms a witness.*

Proof. The proof follows by setting the weight function to be 0 in the proof of Theorem 2.

This shows that in order to show G is k-close or strongly-k-close, we just have to show it is k-close or strongly-k-close for all block matroids of rank $k+1$, respectively.

5 *k*-closeness and isolation

We discuss progress towards conjecture Conjecture 1 in this section. We will prove in Theorem 4 that whenever a certain natural additive combinatorics conjecture is true, any finite abelian group G is $(|G| - 1)$-close. As we noted before, $D(G) \le |G|$, so there is still a gap with conjectured $(D(G)-1)$-closeness for most groups. But the gap will be closed when G is a cyclic group where $D(G) = |G|$.

First, we set up some notions we use later. We say a base B is *isolated* under label ℓ, if it is the unique base with the label $\ell(B)$. A labeling is called *block* *isolating* if it isolates a base whose complement is also a base, i.e. it isolates a block. Next proposition shows that isolation and k-close are related concepts.

Proposition 2. *If every G-labeling of a rank $k+1$ block matroid is not block-isolating, then G is k-close.*

Proof. Suppose not. By Corollary 1, there exists a G-labeling ℓ, $g \in G$ and witness A, B that are bases of a rank $k+1$ block matroid M such that B is the closest g-base to A. Then, B is the unique base that has labeling g. Indeed, if there is any other base $B' \neq B$ with $\ell(B') = g$, then $d(A, B') < k+1 = d(A, B)$, since the block structure guarantees B is the only farthest base from A, contradiction.

Equipped with Proposition 2, our goal become showing block-isolating labeling cannot exists.

5.1 Congruency-constrained base with prime modulus

We will start by proving that Conjecture 1 is true for any cyclic group of prime order, or equivalently congruency-constraints modulo primes. This is a special case of results for general groups which will be introduced in the next subsection. But the proof for cyclic groups of prime order is much simpler followed by a counting argument. For the sake of helping readers gain intuition, we present it as well. The main tool we are going to use is the following additive combinatorics lemma.

Lemma 3 (Schrijver-Seymour[22]). *Let $M = (E, \mathcal{I})$ be a matroid with rank function r and let $\ell : E \to \mathbb{Z}_p$ for some prime number p. Let $E(g) = \ell^{-1}(g)$. Let $\ell(M) := \{\ell(B) \mid B \text{ is a base of } M\}$. Then, $|\ell(M)| \geq \min\{p, \sum_{g \in \mathbb{Z}_p} r(E(g)) - r(M) + 1\}$.*

The following lemma states an exchange property for matroids.

Lemma 4. *Given a matroid $M = (E, \mathcal{I})$, let A be a base, $A_1 \subseteq A$, and B_1 be an independent set such that $A_1 \cap B_1 = \emptyset$. If $|A_1| + |B_1| - r(A_1 \cup B_1) \geq t$, then there exists some $A_2 \subseteq A_1$ and $B_2 \subseteq B_1$ with $|A_2| = |B_2| = t$, such that $A - A_2 + B_2$ is a base.*

Proof. By submodularity of matroid rank functions,

$$r((A \setminus A_1) \cup B_1) + r(A_1 \cup B_1) \geq r(A \cup B_1) + r(B_1) = r(M) + |B_1|.$$

By assumption, $|A_1| + |B_1| - r(A_1 \cup B_1) \geq t$. Combining these, we have

$$r((A \setminus A_1) \cup B_1) \geq r(M) - |A_1| + t = |A \setminus A_1| + t.$$

Using the fact that $A \setminus A_1$ is independent, we deduce there exists $B_2 \subseteq B_1$ with $|B_2| = t$, such that $(A \setminus A_1) \cup B_2$ is independent. Moreover, since A is a base, by adding elements from A_1, $(A \setminus A_1) \cup B_2$ can be extended to a base which has the form $A - A_2 + B_2$ for some $A_2 \subseteq A_1$ with $|A_2| = |B_2| = t$.

Theorem 3. *For any prime p, $\mathbb{Z}_p$ is $(p-1)$-close.*

Proof. Suppose not. Then, by Proposition 2, there exists a block matroid $M = (E, \mathcal{I})$ with $E = A \cup B$, where A and B are two disjoint bases with $|A| = |B| = p$, such that A is isolated under some $\mathbb{Z}_p$-labeling ℓ.

We claim that for any $g \in G$, $r(E(g)) = |E(g) \cap A| + |E(g) \cap B| = |E(g)|$.

Otherwise, letting $A_1 = E(g) \cap A$, $B_1 = E(g) \cap B$ and $t = 1$ in Lemma 4, we can find $a \in A_1$ and $b \in B_1$ such that $A - a + b$ is a base. Since $\ell(a) = \ell(b) = g$, $\ell(A - a + b) = \ell(A)$, contradicting to the fact that A is isolated.

Take any element $e \in B$ and consider matroid $M' = M \setminus e$. By Lemma 3,

$$\begin{aligned}
|\ell(M')| &\geq \min\Big\{p, \sum_{g \in \mathbb{Z}_p} r(E(g)) - r(M) + 1\Big\} \\
&= \min\Big\{p, \sum_{g \in \mathbb{Z}_p, g \neq \ell(e)} r(E(g)) + r(E(\ell(e))) - r(M') + 1\Big\} \\
&= \min\Big\{p, \sum_{g \in \mathbb{Z}_p, g \neq \ell(e)} |E(g)| + (|E(\ell(e))| - 1) - r(M) + 1\Big\} \\
&= \min\Big\{p, \sum_{g \in \mathbb{Z}_p} |E(g)| - p\Big\} = \min\{p, 2p - p\} = p.
\end{aligned}$$

Thus, there exists a base D of M' such that $\ell(D) = \ell(A)$. By definition, D is also a base of M. Since $e \in A$ but $e \notin D$, D is distinct from A, contradicting to the fact that A is isolated under label ℓ.

5.2 General group constraints

To extend beyond prime modulus, we introduce following conjecture of Schrijver-Seymour which is an extension of Lemma 3 to any abelian group G.

Conjecture 3 (Schrijver-Seymour[22], see also [13]). Let $M = (E, \mathcal{I})$ be a matroid with rank function r and let $\ell : E \to G$ for some abelian group G. Let $H = \mathrm{stab}(\ell(M)) := \{h \in G \mid h + \ell(M) = \ell(M)\}$ be the stabilizer of $\ell(M)$. For $Q \subseteq G$, let $E(Q) = \ell^{-1}(Q)$. Then, $|\ell(M)| \geq |H| \cdot \min\big\{\sum_{Q \in G/H} r(E(Q)) - r(M) + 1, |G|/|H|\big\}$.

For a prime p and cyclic group $G = \mathbb{Z}_p$, if $\ell(M) \neq \{0\}$ or $\mathbb{Z}_p$, then it is easy to see $\mathrm{stab}(\ell(M)) = \{0\}$. Then, the inequality in Conjecture 3 reduces to $\ell(M) \geq \min\big\{\sum_{g \in G} r(E(g)) - r(M) + 1,\ |G|\big\}$, which is precisely the form in Lemma 3. Otherwise, $\mathrm{stab}(\ell(M)) = \mathbb{Z}_p$ and the iequality trivially holds.

Theorem 4. *If Conjecture 3 is true for an abelian group G and all of its subgroups, then G is $(|G|-1)$-close.*

Proof. Let $n := |G|$. We prove by induction on n. The theorem trivially holds when $n = 1$. Suppose the theorem does not hold for some $|G| = n$. Then by

Proposition 2, there exists a block matroid $M = (E, \mathcal{I})$, $E = A \cup B$, where A and B are two disjoint bases of M with $|A| = |B| = n$, such that A is isolated under some labeling ℓ.

Take any $e \in A$, let $M' = (E', \mathcal{I}') := M \setminus e$. Let H be the stabilizer of $\ell(M')$. Since G is abelian, H is a normal subgroup of G. Denote by $gH := \{g + h \mid h \in H\}$ the coset of H with representative g. First, observe that if $g \in \ell(M')$, then $gH \subseteq \ell(M')$. This is because, by definition of H, for any $h \in H$, $g + h \in \ell(M')$. Thus $g + H \subseteq \ell(M')$. Therefore, $\ell(M') = \cup_{g \in R}\, gH$, where R is a collection of representatives of cosets of H. It follows that $|\ell(M')| = |R| \cdot |H|$. Let $E'(gH) = E(gH) \setminus e$ for any $g \in G$. Since A is isolated under ℓ and A is not a base of M', we know $\ell(A) \notin \ell(M')$. Thus, $|\ell(M')| < |G|$. Therefore, we can always assume $|\ell(M')| \geq |H|\ (\sum_{g \in R} r(E'(gH)) - r(M') + 1)$ in Conjecture 3. Since $\cup_{g \in R} E'(gH) = E'$ and by submodularity of matroid rank functions, we have $\sum_{g \in R} r(E'(gH)) - r(M') \geq 0$. This implies $|\ell'(M)| \geq |H|$ and thus $|G| > |H|$. It follows that $|R| \geq \sum_{g \in R} r(E'(gH)) - r(M') + 1 = \sum_{g \in R} r(E'(gH)) - n + 1$. Thus,

$$
\begin{aligned}
&\sum_{g \in R} \Big(|A \cap E'(gH)| + |B \cap E'(gH)| - r(E'(gH)) \Big) \\
=&(2n - 1) - \sum_{g \in R} r(E'(gH)) \\
\geq&(2n - 1) - (n + |R| - 1) \\
=&n - |R|.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\max_{g \in R} \Big(|A \cap E'(gH)| + |B \cap E'(gH)| - r(E'(gH)) \Big) \\
\geq&\frac{n - |R|}{|R|} \\
>&\frac{\ell(M') - |R|}{|R|} \\
=&|H| - 1,
\end{aligned}
$$

where the strict inequality follows from $|\ell(M')| < |G|$. Suppose the maximum is attained at g_0. Then, we have

$$
|A \cap E'(g_0H)| + |B \cap E'(g_0H)| - r(E'(g_0H)) \geq |H|,
$$

since $|A \cap E'(g_0H)| + |B \cap E'(g_0H)| - r(E'(g_0H))$ is an integer. Let $A_1 := A \cap E'(g_0H)$, $B_1 := B \cap E'(g_0H)$. By Lemma 4, there exists a base $A - A_2 + B_2$ such that $A_2 \subseteq A_1$, $B_2 \subseteq B_1$ and $|A_2| = |B_2| = |H|$.

Let $M'' = (E'', \mathcal{I}'')$ be the matroid obtained by contracting $A \setminus A_2$ and deleting $B \setminus B_2$, i.e. $M'' = M/\big(A \setminus A_2\big) \setminus \big(B \setminus B_2\big)$. Note that M'' is a rank $|H|$ block matroid with blocks A_2 and B_2. Consider an H-labeling on $E'' = A_2 \cup B_2$, $\ell'' : E'' \to H$, such that $\ell''(e) = \ell(e) - g_0 \in H, \forall e \in E''$. Note that $|H| < |G|$. By the induction hypothesis and the assumption that Conjecture 3 is true for the subgroup H of G, we know that there is no isolating H-labeling of M''. Thus, there must be another base $D_2 \neq A_2$ of M'', such that $\ell''(D_2) = \ell''(A_2)$. Thus $\ell(D_2) = \ell''(D_2) + |H| \cdot g_0 = \ell''(A_2) + |H| \cdot g_0 = \ell(A_2)$. By the definition of contraction, $D := D_2 \cup (A \setminus A_2)$ is a base of matroid M which is distinct from A. And $\ell(D) = \ell(D_2) + \ell(A \setminus A_2) = \ell(A_2) + \ell(A \setminus A_2) = \ell(A)$, contradicting to the fact that A is isolated under label ℓ.

The following lemma is the current status showing on which group the Schrijver-Seymour conjecture is true.

Lemma 5 ([13]). *Conjecture 3 is true for $|G| = pq$ and $G = \mathbb{Z}_{p^n}$ for any p, q prime and n positive integer.*

Finally, $D(G) = |G|$ for cyclic groups [20,2]. This gives us the desired $(D(G)-1)$-closeness for those groups.

Corollary 2. *G is $(D(G) - 1)$-close if $G = \mathbb{Z}_m$, where $m = pq$ or $m = p^n$ for any p, q primes and n a positive integer.*

6 Strongly-k-closeness

This section discusses our results that check if G is strongly-k-close. We have much less understanding of strongly-k-close properties.

6.1 Strong-block-isolation

There is a similar approach to strongly-k-close through strong block-isolation. Given a block matroid, recall a base B is a block if its complement is also a base. A G-labeling *strongly isolates* a block B, if it is the unique block with label $\ell(B)$. A labeling is strongly block-isolating, if it strongly isolates a block.

Proposition 3. *If every G-labeling of a rank $k+1$ block matroid is not strongly-block-isolating, then G is strongly-k-close.*

Proof. Suppose not. By Theorem 2, there exists a G-labeling ℓ, $g \in G$ and witness A, B that are bases of a rank $k + 1$ block matroid M such that B is the closest optimum g-base to A. Fix such labeling ℓ. Since ℓ is not strongly-block-isolating, there is an non-empty $A_1 \subseteq A$ and $B_1 \subseteq B$ such that $A - A_1 + B_1$ is a block with the same label as A. Let $A_2 = A \setminus A_1$, $B_2 = B \setminus B_1$. Then the complement of $A - A_1 + B_1$ is the base $A - A_2 + B_2$, and has the same label as B. Since $w(B) = w(A) - w(A_1) + w(B_1) - w(A_2) + w(B_2) \geq w(A) - w(A_2) + w(B_2) = w(A - A_2 + B_2)$, $A - A_2 + B_2$ is a closer optimum g-base to A, contradiction.

However, it is much harder to reason about strongly-block-isolating labeling, as we don't have a good way of capture blocks. Therefore we only have results either by restricting the matroids, or the sizes of the groups.

6.2 Strongly base orderable matroids

But we do know that Conjecture 2 is true for strongly base orderable matroids. We prove that every G-labeling is strongly-$(D(G)-1)$-close for strongly base orderable matroids in this section.

Theorem 5. *Every G-labeling is strongly-$(D(G)-1)$-close for strongly base orderable matroids.*

Proof. Suppose not. By notng that strongly base orderability is closed under taking minors [15], it is not hard to see from Theorem 2 that the we can find a counterexample which is a rank $D(G)$ strongly orderable block matroid M with blocks A and B as a witness. Let ℓ be a G-labeling of M, w be a weight function such that A is an optimum base and B is the optimum g-base closest from A.

Let bijection $f : B \to A$ be the one satisfying the strongly base orderable property. Define $\ell'(X) = \ell(f(X)) - \ell(X)$ for all $X \subseteq B$. Since $|B| = D(G)$, by the definition of Davenport's constant, there is $B_1 \subseteq B$ such that $\ell'(B_1) = 0$. Let $A_1 = f(B_1)$. This implies $\ell(A_1) = \ell(B_1)$ and thus the base $B - B_1 + A_1$ has the same label as B, i.e. $B - B_1 + A_1$ is also a g-base.

Now, consider $A_2 = A - A_1$ and $B_2 = B - B_1$. One has $f(B_2) = A_2$ and therefore $B - B_2 + A_2 = A - A_1 + B_1$ is also a base. Since A is an optimum base, $w(A - A_1 + B_1) \geq w(A)$, which means $w(A_1) \leq w(B_1)$. Therefore, $w(B - B_1 + A_1) \leq w(B)$. This means $B - B_1 + A_1$ is also an optimum g-base. But $A_1 \subsetneq A$ since $\ell(A) \neq \ell(B)$, which means $B - B_1 + A_1$ is an optimum g-base closer to A, contradiction.

In fact, we can define a weaker property. Two bases A, B are k-replaceable if there exists a bijection $f : B \setminus A \to A \setminus B$ such that for any subset $B' \subseteq B \setminus A$ of size at most k, $B - B' + f(B')$ is a base. We say a matroid is k-replaceable if every pair of bases are k-replaceable. We have Theorem 5 also holds for $(D(G)-1)$-replaceable matroids.

6.3 Small groups

Proposition 3 shows one only has to check block matroids of rank $D(G)$ to know if G is strongly-$(D(G)-1)$-close, we use computers to test out $\mathbb{Z}_3, \mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{Z}_4$, and the results show Conjecture 2 is true for them.

To be more precise, for $G \in \{\mathbb{Z}_3, \mathbb{Z}_2 \times \mathbb{Z}_2\}$, we show that every G-labeling of a rank 3 block matroid is not strongly-block-isolating. Since all but graphic matroid on K_4 are strongly base orderable, so we only have to test a single matroid. Similarly, we tested each $\mathbb{Z}_4$-labeling of a rank 4 block matroid. There are 940 non-isomorphic rank 4 matroids [16], and check all of them is still runs in reasonable amount of time.

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