Champion Spiders in the Game of Graph Nim

Neil J. Calkin* Janine E. Janoski* Allison Nelson† Sydney Ryan‡ Chao Xu§

Abstract

In the game of Graph Nim, players take turns removing one or more edges incident to a chosen vertex in a graph. The player that removes the last edge in the graph wins. A spider graph is a champion if it has a Sprague-Grundy number equal to the number of edges in the graph. We investigate the the Sprague-Grundy numbers of various spider graphs when the number of paths or length of paths increase.

1 Introduction

Nim is an impartial two-player game traditionally played with multiple piles of sticks or stones.

Definition An *impartial game* is a game in which at any given state state of the game, each player has the same set of possible moves [5].

On a player's turn, one or more stones are removed from a pile, and the player that removes the last stone (or stones) is the winner [5]. We want to study a variation of this game, called Graph Nim. In Graph Nim, players remove edges incident to a vertex and the player to remove the last edge(s) is the winner [1].

Definition A *move* in a game of Graph Nim is the removal of one or more edges adjacent to a single vertex.

In particular, we are interested in patterns and periodic behavior of the Sprague-Grundy number, or nimber, as we vary the graph that we are playing on.

Definition The *minimal excluded value*, or *mex*, of a set of integers is the smallest non-negative integer not included in the set.

Definition The *followers* of a graph G are the set of all graphs that can be reached in one move on G. We denote the followers of G as $F(G)$.

Using the previous definitions, we can define the Sprague-Grundy number as follows:

Definition Let G be a graph. We define a function $g(G) = \text{mex}(\{g(H)|H \in F(G)\})$. We call $g(G)$ the *Sprague-Grundy number*, or *nimber* of G. We denote $\{g(H)|H \in F(G)\} = gF(G)$.

The Sprague-Grundy Theorem allows us to assume that games of Graph Nim actually have a Sprague-Grundy number.

Theorem 1.1 *(Sprague-Grundy Theorem) Every impartial game under the normal play convention is equivalent to a nimber.*

We have the following theorem for disjoint games.

Theorem 1.2 *The Sprague-Grundy number of a game consisting of disjoint components is the nim-sum of the Sprague-Grundy numbers of those components [3].*

*Clemson University
†Appalachian State University
‡Grinnell College
§Stony Brook University

Definition We define the *nim-sum* of two non-negative integers as binary addition with no carries. We denote it as $\oplus$ [2].

We have shown that for certain classes of graphs the nimbers become periodic as we make the graphs larger.

Definition A *spider graph* is a graph with one vertex of degree greater than 2 and all other vertices with degree at most 2. We will extend the class of spider graphs to include paths for our purposes.

For example, the graph in Figure 1 is a spider graph.

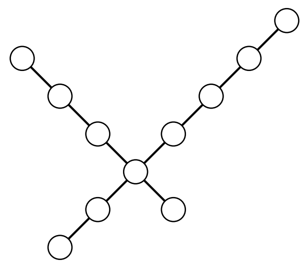

Figure 1.1. A spider graph

By studying spider graphs, we describe methods for determining the nimbers of several different types of spiders used in a game of Graph Nim. We are interested in the periodicity of the nimbers as we increase the length and number of paths attached to the spider. The Sprague-Grundy number of a game analyzes whether or not the "first player" has a winning strategy. A non-zero nimber ensures that the first player will always win if he plays the correct strategy. A nimber of zero means that the first player does not have a winning strategy and will always lose if his opponent plays the correct strategy [2].

To find the Sprague-Grundy number of a graph G, we want to find the minimal excluded value out of the Sprague-Grundy numbers of all graphs that can be obtained in one move on G.

For example, the graph in Figure 1.2 has the followers in Figure 1.3.

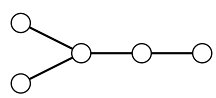

Figure 1.2. The graph G

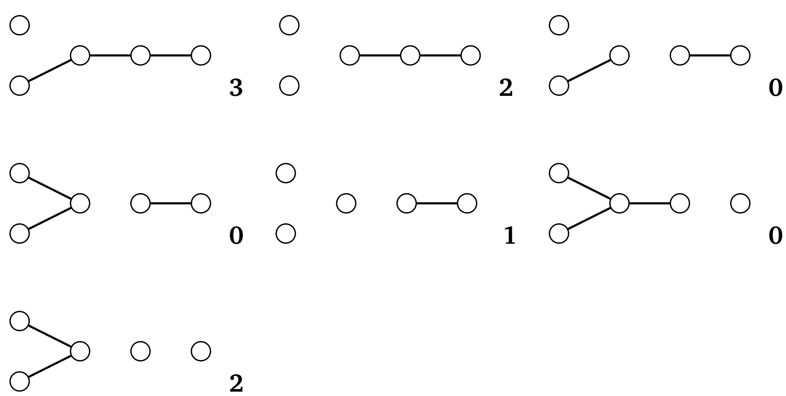

Figure 1.3. Followers of G and their S-G Numbers

The nimber for the graph in Figure 1.2 is $g(G) = \text{mex}(\{3,2,1,0\}) = 4$.

The following result bounds the Sprague-Grundy number for any graph.

Theorem 1.3 *$g(G) \leq e(G)$ where G is a graph and $e(G)$ is the number of edges in G.*

Proof: Proof by contradiction.
Let G be a counterexample with a minimal number of edges. A graph with no edges has a nimber of 0, thus it can't be a counterexample, and thus $e(G) > 0$. This means $g(G) > e(G)$, so there must be a follower $G' \in F(G)$ whose nimber is $e(G)$. $e(G) > e(G')$, since G' is a follower of G. This means that $g(G') > e(G')$, which makes G' a counterexample as well. We assumed that G was the minimal counterexample, so this is a contradiction. Thus $g(G) \leq e(G)$. □

From now on we will use $e(G)$ to denote the number of edges in G.
We can think of a spider as consisting of a central vertex, or a body, and a set of paths adjacent to it, the legs.

Definition The *body* of a spider graph is the vertex of degree > 2.

Definition The *legs* of a spider graph are the paths connected to the body.

There are two distinct types of moves in Graph Nim on spider graphs, we call them stomps and cuts.

Definition A *stomp* removes one or more edges adjacent to the body.

Definition A *cut* removes edges not adjacent to the body.

We want to study spider graphs that are somehow related to see if there is any pattern to their Sprague-Grundy numbers. To do this, we consider a mapping between spider graphs and integer partitions. If a spider graph consists of a_1 legs of length 1, a_2 legs of length 2, ..., a_n legs of length n, we represent this graph by the partition

$$(\underbrace{n, \ldots, n}_{a_n \text{ times}}, \underbrace{n-1, \ldots, n-1}_{a_{n-1} \text{ times}}, \ldots, \underbrace{1, \ldots, 1}_{a_1 \text{ times}}).$$

We have created programs in both Java and Sage to calculate the nimbers of various graphs represented as partitions and output them in lists that we can sort and analyze. After looking through the data in the Appendix, a few simple patterns were observed.

2 Sprague-Grundy Numbers of Spider Graphs

First, we consider spider graphs with k legs of length 1.

Definition A *k-star* is a graph consisting of a single vertex and k paths of length 1 adjacent to the vertex. We denote a k-star as S_k.

Theorem 2.1 *For all $k \geq 1$, $g(S_k) = k$ where S_k is a star graph with k edges [4].*

Let P_k denote a path of length k. Denote $\underbrace{P_k \cup \ldots \cup P_k}_{n \text{ times}}$ by $(n)P_k$. The Sprague-Grundy numbers for paths are known [1]. The Sprague-Grundy numbers become periodic as the length of the path grows larger than 72. We can use the path nimbers to help analyze simple spider graphs.

Lemma 2.2 *$x - y \leq x \oplus y \leq x + y$ for $x, y \in \mathbb{N}$.*

Proof: Since $\oplus$ is binary addition without carries, $x \oplus y \leq x + y$. Assume $x \oplus y < x - y$ for some x and y. Then

$$x = x \oplus (y \oplus y) = (x \oplus y) \oplus y < (x - y) \oplus y \leq (x - y) + y = x.$$

This is a contradiction. Thus, $x - y \leq x \oplus y$. □

Lemma 2.3 *For $a, b, k \in \mathbb{N}$, $[a + k..b - k] \subseteq k \oplus [a..b]$.*

Proof: Let $x \in [a + k..b - k]$. We can write $t \oplus k = x$. Then $x \oplus k = t$ and $x - k \leq t \leq x + k$ by Lemma 2.2. Thus $t \in [(a + k) - k..(b - k) + k] = [a..b]$ so $x \in k \oplus [a..b]$. □

Lemma 2.4 *For $b, k \in \mathbb{N}$, $[0..b - k] \subseteq k \oplus [0..b]$.*

Proof: Let $x \in [0..b - k]$. We can write $t \oplus k = x$, so $x \oplus k = t$. This implies $t \in [0 - k..(b - k) + k]$. Since $t \geq 0$, we must have that $t \in [0..b]$. Thus $x \in k \oplus [0..b]$. □

3 Champion Spiders

Definition A *champion spider* is a spider graph whose Sprague-Grundy number is equal to its number of edges. In other words, $g(n^{a_n}(n-1)^{a_{n-1}}\dots 1^{a_1}) = n(a_n)+(n-1)(a_{n-1})+\dots+a_1$ where n is the maximum leg length.

Theorem 3.1 $g(2^{a_2}1^{a_1}) = 2a_2 + a_1$ *if* $a_1 \geq 2a_2 - 2$ *for* $a_1,\ a_2 \in \mathbb{N}$.

Proof: Proof by induction on a_2.

Base case: $a_2 = 0$. $g(2^0 1^{a_1}) = a_1$ for all a_1 by Theorem 2.1.

Inductive Step:

Assume the hypothesis is true for all integers less than a_2. Consider $2^{a_2}1^{a_1}$, where $a_1 \geq 2a_2 - 2$. The followers of $2^{a_2}1^{a_1}$ are as follows:

1. $2^{a_2-1}1^{a_1+1}$
2. $2^{a_2-1}1^{a_1}$
3. $2^j1^k \cup (a_2-j)P_1$ for all $j \leq a_2, k \leq a_1$ and $j+k < a_2+a_1$. Note:

$$g(2^j1^k \cup (a_2-j)P_1) = \begin{cases} g(2^j1^k \cup P_1) & \text{if } a_2 - j \equiv 1 \mod 2 \\ g(2^j1^k) & \text{if } a_2 - j \equiv 0 \mod 2. \end{cases}$$

Claim: $\{0,\dots,2a_2+a_1-1\} \subseteq gF(2^{a_2}1^{a_1})$. For any $h \in \{0,\dots,2a_2+a_1-1\}$, there are three cases:

Case 1 $h = 2a_2 + a_1 - 1 = g(2^{a_2-1}1^{a_1+1})$ by induction.

Case 2 $h = 2a_2 + a_1 - 2 = g(2^{a_2-1}1^{a_1})$ by induction.

Case 3 $h < 2a_2 + a_1 - 2$. Note that we can write $h = 2j + k$ where $0 \leq j \leq a_2 - 1$ and $2j - 2 \leq k < a_1$. We will show that there exists a follower graph with nimber equal to h for all h in this range.

1. If $a_2 - j$ is even, then $g(2^j1^k) = 2j + k = h$ by hypothesis.
2. If $a_2 - j$ is odd and h is odd, then k is odd. Therefore $k > 2j - 2$ and $k - 1 \geq 2j - 2$, so

$$g(2^j1^{k-1} \cup P_1) = (h-1) \oplus 1 = h.$$

3. If $a_2 - j$ is odd and h is even. $k < a_1$ thus $k + 1 \leq a_1$, so

$$g(2^j1^{k+1} \cup P_1) = (h+1) \oplus 1 = h.$$

This shows $\{0,\dots,2n+a_1-1\} \subseteq gF(2^{a_2}1^{a_1})$, and because $g(2^{a_2}1^{a_1}) \leq 2a_2+a_1$, we have $g(2^{a_2}1^{a_1}) = 2a_2+a_1$. □

Definition The *integer interval* between a and b is the set of consecutive integers between and including a and b, denoted $[a..b]$.

Definition The numbers $[a..k]$, $a \geq 0$, are *generated* by a graph G if there exists a follower of G whose Sprague-Grundy number equals c for all $c \in [a..k]$.

We denote $k \oplus [a..b] = \{k \oplus x | x \in [a..b]\}$. Similarly, we denote $k + [a..b] = \{k + x | x \in [a..b]\}$.

4 Stability

Definition Define the operation + between two spiders to be

$$n^{a_n}\dots 1^{a_1} + n^{b_n}\dots 1^{b_1} = n^{a_n+b_n}\dots 1^{a_1+b_1}$$

Note that this is a vertex contraction on the two root vertices and that it is is associative and commutative.

Definition The spider G is said to be *stable* if all spiders of the form $G + 1^k$ where $k \geq 0$ are champion spiders.

In other words, if a spider is stable then we can add any number of legs of length 1 to the spider and the resulting spider is champion.

Definition Let $f(G)$ be the smallest natural number such that $G + 1^{f(G)}$ is stable. If no such number exists, it is defined as ∞.

4.1 Algorithm for finding $f(G)$

In order to find an algorithm to calculate $f(G)$, we need an algorithm to decide if $G+1^{a_1}$ is stable. The following theorem provides a method for determining a graph's stability by checking only a finite number of cases. This implies that an algorithm to find $f(G)$ exists.

Theorem 4.1 *Let G be a spider. If $G+1^{a_1}$ is champion for $0 \le a_1 \le d$ where $d = 13+e(G)$, then G is stable.*

Proof: To show that $G+1^{a_1}$ is stable, we must show that $G+1^{a_1}$ is champion for all $a_1 \ge 0$. We induct on a_1.

Base case We have that $G+1^{a_1}$ is champion for $0 \le a_1 \le d$ by the inductive hypothesis.

Inductive Step Assume the hypothesis is true for all $d < a_1 < m$. Consider the case when $m = a_1$. In order to show $G+1^{a_1}$ is a champion, we have to show $gF(G+1^{a_1}) = [0..e(G+1^{a_1})-1]$.

We have

$$\{g(G+1^k) | 0 \le k < a_1\} = [e(G)..e(G+1^{a_1-1})] = [e(G)..e(G+1^{a_1})-1]$$

and by Lemma 5.3, we know

$$gF(G+1^{a_1}) \supseteq [0..a_1-15] \supseteq [0..d+1-15] = [0..14+e(G)-15] = [0..e(G)-1].$$

Therefore, we can see that $G+1^{a_1}$ is champion. □

Using Theorem 4.1, we have an algorithm, Stable(G), to check if G is stable.

```
stable(G):
  bound ← 13 + e(G)
  for k from 0 to bound:
      if e(G + 1^k) ≠ g(G + 1^k):
          return false
  return true
```

An algorithm for computing $f(G)$ follows by checking if $G+1^k$ is stable for every k. If the algorithm terminates, then we found $f(G)$. Otherwise $f(G) = \infty$. Note that we have no bounds on $f(G)$, and thus there is no way to determine that $f(G) = \infty$.

```
f(G):
  for k from 0 to ∞:
      if stable(G + 1^k):
          return k
```

Example This algorithm calculates $f(3^1) = 2$. That is, when $a_1 \ge 2$, $3^1 1^{a_1}$ is champion, and $3^1 1^{2-1}$ is not a champion. These values can be verified by observing $3^1 1^k$ is a champion for $k \in [2..2+3+1] = [2..6]$. Similarly, it finds $f(4^1) = 4$ by checking $4^1 1^k$ is a champion for $k \in [4..4+4+1] = [4..9]$.

4.2 A result about champions, $f(G)$ and $f(G+2^1)$

Theorem 4.2 *For all spiders G,*

$$f(G+2^1) \le f(G) + e(G) + 15$$

Proof: For convenience, let $N = e(G) + a_1 + 2$.

If $f(G) = \infty$ then the theorem is trivially true.

Suppose $f(G)$ is finite, and let $c' = f(G) + (N-2-a_1) + 15$. We want to show for any $a_1 \ge c'$, $G+2^1+1^{a_1}$ is a champion. If $gF(G+2^1+1^{a_1}) = [0..N-1]$, then $G+2^1+1^{a_1}$ is a champion. Consider the following three cases that will generate all the nimbers in the range.

Case 1 The follower graphs $G+1^{a_1+1}$ and $G+1^{a_1}$ have the nimbers $N-1$ and $N-2$ respectively.

Case 2 Consider the set of graphs $X = \{G+1^{f(G)} \cup P_1, G+1^{f(G)+1} \cup P_1, \ldots, G+1^{a_1} \cup P_1\}$. Each one is a follower of $G+2^1+1^{a_1}$. Note that all of these graphs are champion spiders union a path of length 1. Thus, we get

$$\begin{aligned}\{g(x)|x \in X\} &= 1 \oplus [f(G)+e(G)..e(G)+2+a_i] \\ &= 1 \oplus [c'-15..N-2] \\ &\supseteq [c'-14..N-3]\end{aligned}$$

by Lemma 2.3.

Case 3 Using Corollary 5.3, we know

$$gF(G+2^1+1^{a_1}) \supseteq [0..a_1-15] \supseteq [0..c'-15]$$

The three sets cover the entire interval $[0..N-1]$, therefore $G+2^1+1^{a_1}$ is a champion for all $a_1 \geq c'$.

$$c' = f(G)+(N-2-a_1)+15 = f(G)+e(G)+15 \geq f(G+2^1)$$

. □

Thus, we have a bound on $f(G+2^1)$ in terms of $f(G)$.

5 Periodicity in the Discrepancy

It turns out, not all spiders G admit a finite $f(G)$, however, they still have periodic behavior. It is captured by the notion of discrepancy.

Definition The discrepancy of a graph G, $d(G)$, is defined as

$$d(G) = e(G) - g(G)$$

Notice that champion spiders are precisely the spiders with discrepancy 0.

Definition If G is a graph, v a vertex of G. Define $G +_v S_k$ as the graph produced by vertex contraction of v and the body of S_k. Let $\{a_i\}$ be a sequence, such that $a_i = d(G +_v S_i)$. If the sequence is eventually periodic, then the period is $p_v(G)$ and starting point is $c_v(G)$.

Theorem 5.1 *Let G be a graph and v a specific vertex on G, and G' is a graph can be obtained by remove all edges of v from G, then $gF(G +_v S_k) \supseteq [0..k-1-g(G')] \supseteq [0..k-1-e(G)]$. If v is not a isolated vertex in G, then $gF(G +_v S_k) \supseteq [0..k-g(G')] \supseteq [0..k-e(G)]$*

Proof: $G' \cup S_j$ is a follower graph for all $j \in [0..k-1]$. Thus we get all values in $g(G') \oplus [0..k-1] \supseteq [0..k-1-g(G')] \supseteq [0..k-1-e(G)]$. When v is not an isolated vertex, $G' \cup S_k$ is also a follower graph, and this give us the second part. □

Corollary 5.2 *Let G be a graph, $v \in V(G)$, then $d(G +_v S_k) \leq 2e(G)$ for all k.*

Corollary 5.3 *For all spiders G, $gF(G +_v S_k) \supseteq [0..k-15]$.*

Proof: If G has at least one edge, then by Theorem 5.1, $G +_v S_k \supseteq [0..a_1-15]$. If G is empty, then $gF(G +_v S_k) = [0..k-1] \supseteq [0..k-15]$. □

Theorem 5.4 *Let G be a graph, $v \in V(G)$, then the sequence $\{a_n\}$ is eventually periodic with starting position $c_v(G)$ and period $p_v(G)$, repectively. Here $a_k = d(G +_v S_k)$.*

Proof: Proof by induction on $e(G)$.

Base Case: $e(G) = 0$, then we have $d(G +_v S_k) = 0$ for all k.

Inductive Step: By inductive hypothesis, we know for all G', where G' is a proper subgraph of G, $c_v(G)$ and $p_v(G)$ are finite.

Let $c = \max(\{c_v(G')|G' \subseteq G\})$ and $p = \text{lcm}(\{p_v(G')|G' \subseteq G\})$. Let $k \geq \max(c, 2e(G))$. Due to Theorem 5.1, it is known that $gF(G +_v S_k) \supseteq [0..k-1-e(G)]$. Thus $d(G +_v S_k)$ is determined solely by followers that have a potential of having nimber in between $[k-e(G)..k+e(G)-1]$.

$G' +_v S_j$ for $G' \subseteq G$ and $j \in [k-2e(G)..k]$ captures all the graphs in that set.

Consider $d(G +_v S_{c+mp})$ and $d(G +_v S_{c+m'p})$ for some natural number m, m'. Because of periodicity, $d(G' +_v S_{c+p}) = d(G' +_v S_{c+mp})$ for all $G' \subseteq G$. So $d(G +_v S_{c+mp})$ is determined by no more than the value $d(G +_v S_{c+mp-e(G)})$ to $d(G +_v S_{c+mp-1})$. Similarly, $d(G +_v S_{c+m'p})$ is determined by the value of $d(G +_v S_{c+m'p-e(G)})$ to $d(G +_v S_{c+m'p-1})$. This shows if there exist some m, m', $m \neq m'$, such that for all $0 \leq i \leq 2e(G)$,

$$d(G +_v S_{c+mp-2e(G)+i}) = d(G +_v S_{c+m'p-2e(G)+i}) \quad (*)$$

then we have a periodic behavior with period $(m'-m)p$ and starting position $c+mp-2e(G)$.

There are at most $(2e(G))^{2e(G)}$ possible values for $d(G +_v S_{c+mp-2e(G)}), d(G +_v S_{c+mp-2e(G)+1}), \ldots, d(G +_v S_{c+mp-1})$, by pigeonhole principle, there exist $m, m' \leq (2e(G))^{2e(G)} + 1$, such that $m \neq m'$ and (*) holds. □

Notice this theorem proves the conjecture B(ii) and C(ii) in [4].

The theorem implies a fast method to find $c_v(G)$ and $p_v(G)$. Running computation on path graphs P_n, we find $p_v(P_n)$ is a power of 2 for any v and for n up to 30.

Conjecture 5.5 *$p_v(G)$ is a power of 2 for all graph G and any vertex v in G.*

This is not suprising, since $\oplus$ is effectively working over a group of 2^n elements when the integer inputs are bounded.

6 Conjectures on Champion Spiders

6.1 $3^{a_3}2^{a_2}1^{a_1}$

Conjecture 6.1 *$3^{a_3}2^{a_2}1^{a_1}$ is a champion spider if $a_2 > 0$ and for all $3^i2^j \subseteq 3^{a_3}2^{a_2}$, $a_1 > f(3^i2^j) + 15$ where $f(3^i2^j)$ is defined in Section 4.*

Conjecture 6.2 *Let $a_3 \geq 10$. For*

$$a_1 \geq \begin{cases} 3a_3 - 7 & \text{if } a_3 \text{ even} \\ 3a_3 - 6 & \text{if } a_3 \text{ odd,} \end{cases}$$

$3^{a_3}1^{a_1}$ is champion.

We formed this conjecture by looking at values of $d(3^{a_3}1^{a_1})$, which are contained in Appendix A. After a certain number of 1s, we noticed that the discrepancy became zero and stayed there for larger values of a_1, indicating that these graphs become stable after a point.

Conjecture 6.2 can be shown true for a given a_3 by applying a refined version of Theorem 4.1. This theorem states that we only need to verify if $d(3^{a_3}1^k) = 0$ for all $3a_3 - 6 \leq k \leq 6a_3 - 5$ if a_3 is odd, and if $d(3^{a_3}1^k) = 0$ for all $3a_3 - 7 \leq k \leq 6a_3 - 6$ if a_3 is even. We have done this for all $a_3 \leq 30$.

In fact, if we call m the number of ones for which the graphs become stable, we found predictable behavior of $d(3^{a_3}1^{m-1}), d(3^{a_3}1^{m-2}), \ldots, d(3^{a_3}1^{m-5})$ for $a_3 \geq 12$. In particular, we have the following conjecture.

Conjecture 6.3 *Let $a_3 \geq 12$. For*

$$a_1 = \begin{cases} 3a_3 - 7 & \text{if } a_3 \text{ even} \\ 3a_3 - 6 & \text{if } a_3 \text{ odd,} \end{cases}$$

$3^{a_3}1^{a_1}$ is stable if and only if we get the following:

For a_3 even,

$$d(3^{a_3}1^{3a_3-8}) = 3$$
$$d(3^{a_3}1^{3a_3-9}) = 0$$
$$d(3^{a_3}1^{3a_3-10}) = 4$$
$$d(3^{a_3}1^{3a_3-11}) = 4$$
$$d(3^{a_3}1^{3a_3-12}) = 8$$

For a_3 odd,

$$d(3^{a_3}1^{3a_3-7}) = 5$$
$$d(3^{a_3}1^{3a_3-8}) = 0$$
$$d(3^{a_3}1^{3a_3-9}) = 0$$
$$d(3^{a_3}1^{3a_3-10}) = 3$$
$$d(3^{a_3}1^{3a_3-11}) = 5$$

Example Let $a_3 = 15$. Then, let $a_1 = 3a_3 - 6 = 39$. If we calculate $d(3^{a_3}1^{a_3-6-i})$ for $i \in [1,2,3,4,5]$, we get the desired values 5,0,0,3,5. This means that $3^{15}1^{39}$ is champion. Alternatively, if we examine a spider of the form 3^i1^j that we know to be champion that satisfies the requirements on i and j, we can look it up in the table and if the conjecture is true, we will see the forementioned pattern in the discrepancies of smaller graphs.

7 Conclusion

We have developed methods for computing the Sprague-Grundy numbers of certain types of spider graphs. Ultimately, our results discuss only a small subset of these graphs. We hope in the future to be able to extend our results using the new methods presented in Section 5. In addition, we plan to further work on the conjectures in Section 6 and show that the graphs $3^{a_3}2^{a_2}1^{a_1}$ become stable. Additionally, we would like to be able to say something about the length of the predictable pattern in values of the discrepancy as we force a_3 to be larger, which we mention after Conjecture 6.3. It is possible that this pattern before acheiving stability could also be generalized to other spider graphs.

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A $d(3^{a_3}1^{a_1})$ Data

The nth column and mth row represents $d(3^{n+9}1^{m-1})$.

	3^{10}	3^{11}	3^{12}	3^{13}	3^{14}	3^{15}	3^{16}	3^{17}	3^{18}	3^{19}	3^{20}	3^{21}	3^{22}	3^{23}	3^{24}	3^{25}	3^{26}	3^{27}	3^{28}	3^{29}	3^{30}
1^{0}	22	25	24	24	27	33	30	33	37	40	40	40	43	49	48	48	51	57	56	56	59
1^{1}	21	20	24	27	29	30	33	33	36	36	40	43	45	44	48	51	53	52	56	59	61
1^{2}	19	17	21	24	26	27	27	30	36	35	35	38	44	43	43	46	52	49	53	56	58
1^{3}	17	17	20	20	23	27	30	28	31	35	38	36	39	43	46	46	49	49	52	54	57
1^{4}	12	15	19	22	20	23	27	30	28	31	35	38	38	38	41	43	46	46	49	49	52
1^{5}	15	15	18	20	23	23	26	28	31	31	34	34	38	41	43	42	46	49	49	52	52
1^{6}	12	12	15	17	20	20	23	23	26	30	33	33	33	36	42	41	41	44	46	49	49
1^{7}	12	15	17	16	20	23	23	26	26	29	29	33	36	38	37	41	44	46	45	46	52
1^{8}	11	17	16	15	19	19	20	26	29	29	29	32	34	37	37	40	40	44	47	45	46
1^{9}	13	11	14	14	18	21	17	21	24	26	26	29	31	31	34	40	37	40	44	47	46
1^{10}	11	14	14	14	17	17	21	24	26	26	29	27	31	34	36	37	40	40	43	43	46
1^{11}	10	10	14	17	17	17	20	26	23	24	27	26	29	29	32	32	35	39	42	40	44
1^{12}	10	10	13	13	17	20	22	23	23	26	25	28	29	32	32	35	35	38	40	40	43
1^{13}	10	13	13	12	16	16	17	23	26	25	28	26	27	27	30	32	31	36	40	43	42
1^{14}	8	8	11	11	15	18	17	20	22	21	25	28	27	30	32	32	35	35	35	38	40
1^{15}	3	7	10	10	13	13	16	18	21	21	24	24	27	28	28	31	37	35	38	36	40
1^{16}	3	6	12	9	13	16	14	17	21	24	24	27	27	28	31	33	32	33	36	35	38
1^{17}	6	8	9	9	12	14	17	17	20	20	23	27	25	28	29	32	32	35	34	37	38
1^{18}	8	5	9	9	8	13	17	20	19	22	20	24	27	25	28	32	35	34	37	35	33
1^{19}	5	5	5	9	12	12	12	15	17	20	20	23	25	28	28	31	31	34	35	34	38
1^{20}	5	5	10	9	8	12	15	13	16	20	23	21	24	28	31	31	34	34	34	38	38
1^{21}	5	4	8	8	8	11	13	16	16	19	21	24	24	27	27	30	34	34	33	37	35
1^{22}	3	0	3	5	8	8	11	13	16	16	19	19	22	26	29	29	29	32	33	36	34
1^{23}	0	3	5	8	8	11	9	12	16	19	19	22	22	25	25	29	32	34	31	34	34
1^{24}	0	0	8	5	4	8	11	11	11	14	16	16	19	25	22	26	29	31	30	34	37
1^{25}	0	0	4	4	4	7	13	11	14	16	16	19	21	22	22	25	31	30	30	33	35
1^{26}	0	5	4	4	7	9	8	8	16	13	13	21	18	22	22	27	26	27	33	31	35
1^{27}	0	0	0	3	9	8	8	11	13	13	13	17	17	22	21	25	28	27	30	28	32
1^{28}	0	0	3	5	4	8	11	13	13	13	18	17	17	20	20	23	23	26	28	28	31
1^{29}	0	0	0	3	0	8	9	9	9	18	17	14	15	22	19	23	26	24	28	31	33
1^{30}	0	0	0	0	8	5	9	9	14	13	14	14	17	19	19	22	24	24	27	33	32
1^{31}	0	0	0	0	4	4	9	8	12	15	14	17	19	19	22	20	24	27	29	28	32
1^{32}	0	0	0	5	4	4	7	7	7	10	12	12	15	17	20	20	23	25	28	28	31
1^{33}	0	0	0	0	0	3	9	7	10	12	12	15	17	16	20	23	21	24	28	31	29
1^{34}	0	0	0	0	3	5	4	7	5	8	8	17	16	16	19	21	24	24	27	29	32
1^{35}	0	0	0	0	0	3	0	6	9	8	8	11	13	16	16	19	21	21	24	26	29
1^{36}	0	0	0	0	0	0	8	5	9	8	11	13	12	16	19	17	21	24	26	25	29
1^{37}	0	0	0	0	0	0	4	4	4	7	13	12	12	15	17	17	20	26	25	25	28
1^{38}	0	0	0	0	0	5	4	4	7	9	8	12	15	13	17	20	22	21	25	28	26
1^{39}	0	0	0	0	0	0	0	3	9	8	8	11	13	13	16	22	21	21	24	26	26
1^{40}	0	0	0	0	0	0	3	5	4	8	11	13	13	16	18	17	21	24	26	26	29
1^{41}	0	0	0	0	0	0	0	3	0	8	9	9	9	18	17	17	20	22	22	25	31
1^{42}	0	0	0	0	0	0	0	0	8	5	9	9	14	13	14	20	18	22	25	25	28
1^{43}	0	0	0	0	0	0	0	0	4	4	9	8	12	15	14	17	17	20	20	23	25
1^{44}	0	0	0	0	0	0	0	5	4	4	7	7	7	10	12	12	15	17	17	20	22
1^{45}	0	0	0	0	0	0	0	0	0	3	9	7	10	12	12	15	17	17	20	22	21
1^{46}	0	0	0	0	0	0	0	0	3	5	4	7	6	9	10	13	13	16	16	20	23
1^{47}	0	0	0	0	0	0	0	0	0	3	0	6	9	9	9	13	16	16	16	19	21
1^{48}	0	0	0	0	0	0	0	0	0	0	8	5	9	9	8	12	12	16	19	17	20
1^{49}	0	0	0	0	0	0	0	0	0	0	4	4	4	7	8	11	9	12	16	19	17
1^{50}	0	0	0	0	0	0	0	0	0	5	4	4	7	9	8	9	12	12	15	17	20
1^{51}	0	0	0	0	0	0	0	0	0	0	0	3	9	8	8	8	12	15	13	16	20
1^{52}	0	0	0	0	0	0	0	0	0	0	3	5	4	8	8	8	11	13	16	16	19
1^{53}	0	0	0	0	0	0	0	0	0	0	0	3	0	8	8	8	8	11	13	16	16
1^{54}	0	0	0	0	0	0	0	0	0	0	0	0	8	5	8	8	11	13	12	16	19
1^{55}	0	0	0	0	0	0	0	0	0	0	0	0	4	4	4	7	13	12	12	15	17
1^{56}	0	0	0	0	0	0	0	0	0	0	0	5	4	4	7	9	8	12	15	13	17
1^{57}	0	0	0	0	0	0	0	0	0	0	0	0	0	3	9	8	8	11	13	13	16
1^{58}	0	0	0	0	0	0	0	0	0	0	0	0	3	5	4	8	11	13	13	16	18
1^{59}	0	0	0	0	0	0	0	0	0	0	0	0	0	3	0	8	9	9	9	18	17
1^{60}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	8	5	9	9	14	13	14
1^{61}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4	4	9	8	12	15	14
1^{62}	0	0	0	0	0	0	0	0	0	0	0	0	0	5	4	4	7	7	7	10	12
1^{63}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	3	9	7	10	12	12
1^{64}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	3	5	4	7	6	9	10
1^{65}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	3	0	6	9	9	9
1^{66}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	8	5	9	9	8
1^{67}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4	4	4	7	8
1^{68}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	5	4	4	7	9	8
1^{69}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	3	9	8	8
1^{70}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	3	5	4	8	8
1^{71}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	3	0	8	8
1^{72}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	8	5	8
1^{73}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4	4	4
1^{74}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	5	4	4	7
1^{75}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	3	9
1^{76}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	3	5	4
1^{77}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	3	0
1^{78}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	8
1^{79}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4
1^{80}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	5	4
1^{81}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1^{82}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	3
1^{83}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
1^{175}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

B $d(n^1 1^k)$

	2^1	3^1	4^1	5^1	6^1
1^0	0	0	3	1	3
1^1	0	3	1	3	5
1^2	0	0	0	5	7
1^3	0	0	0	3	9
1^4	0	0	3	1	1
1^5	0	0	0	3	0
1^6	0	0	0	5	0
1^7	0	0	0	3	0
1^8	0	0	0	0	0
1^9	0	0	0	3	5
1^{10}	0	0	0	5	0
1^{11}	0	0	0	0	9
1^{12}	0	0	0	0	0
1^{13}	0	0	0	3	0
1^{14}	0	0	0	5	3
1^{15}	0	0	0	0	0
1^{16}	0	0	0	0	0
1^{17}	0	0	0	3	0
1^{18}	0	0	0	5	0
1^{19}	0	0	0	0	0
1^{20}	0	0	0	0	0
1^{21}	0	0	0	3	0
1^{22}	0	0	0	5	0
1^{23}	0	0	0	0	0
1^{24}	0	0	0	0	0
1^{25}	0	0	0	3	0