Global and fixed-terminal cuts in digraphs

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What to expect

- Survey talk.
- Lot of definitions, problems and examples.
- Few technical details.
- Many open problems.

Let $G = (V, E)$ be a graph.

A cut (node cut) is a set of edges (nodes) that disconnects some pair of vertices if removed. The value of a cut (node cut) is the number of edges (nodes) in the cut.

min global cut (node cut) problem:

- Input: G.
- Output: Minimum (node) cut.

min local cut (node cut) problem:

- Input: $G, s, t \in V$.
- Output: Minimum (node) cut that disconnects s and t.

Complexity separation of local and global cuts

- local cut at least as hard as global cut.
- min cut can be reduced to $O(n^2)$ calls to min local cut.
- Main problem: When is local cut strictly harder than global cut?
  - Local cut is NP-hard but polynomial time algorithm for global cut.
  - α-inapproximability of local cut, but $\alpha - \delta$-approximation for global cut for $\delta > 0$.

Local and global cuts in undirected graphs

Cuts in undirected graphs

Consider a undirected graph $G = (V, E)$. We say $s, t \in V$ are disconnected if there is no paths between s and t.

- cut: A set of edges s.t. its removal disconnects some pair of vertices.
- st-cut: A set of edges s.t. its removal disconnect s and t.
- Each set $S \subset V \setminus \{t\}$ such that $s \in S$ determines a st-cut with $|\delta(S)|$ edges.
- $\lambda(s, t; G)$ is the value of min st-cut.
- $\lambda(G) = \min_{s,t \in V} \lambda(s, t; G)$ is the value of min cut.

Algorithmic aspects

- $\lambda(s,t) = \min_{s \in S} |\delta(S)|$ can be computed in $O(nm)$ time by reducing to maximum flow. [Orlin 2013]
- $\lambda(G)$ can be computed directly using MA-ordering. [Nagamochi & Ibaraki 1992]

Node cuts

- $\kappa(s, t; G)$ is the value of min st-node-cut.
- $\kappa(G) = \min_{s,t \in V} \kappa(s, t; G)$ is the value of min node-cut.

Both can be solved in polynomial time, since $\kappa(s, t; G)$ reduces to maximum flow.

No complexity separation

Both global and local problem for edge and node deletion can be solved in polynomial time.

Local and global cuts in digraphs

Two definitions of st-cut

For a graph G and two vertices s and t. A st-cut is a set of edges E', s.t. in $G - E'$

- Definition 1: There is no path between s and t.
- Definition 2: There is no vertex that can reach both s and t.

Two notions are the same in undirected graphs.

Bicut and double cut

Consider a digraph $G = (V, E)$.

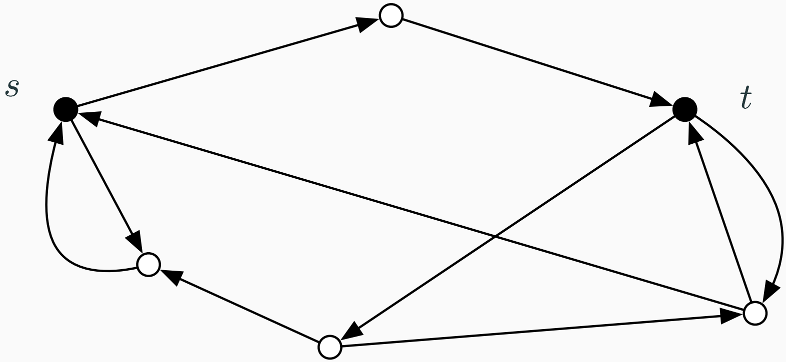
Definition (st-bicut)

A set of edges E' is a st-bicut, if there is no path between s and t in $G - E'$

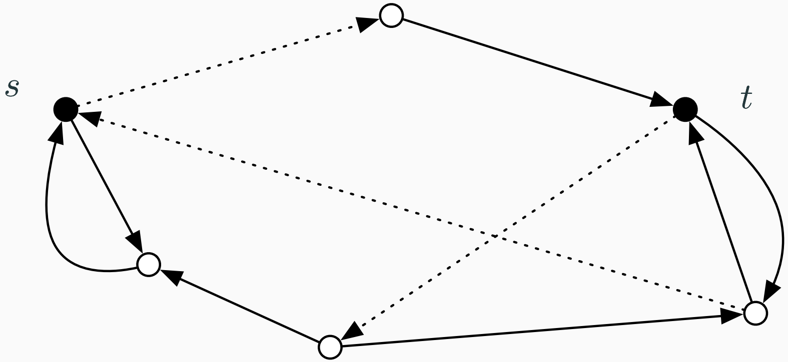
Definition (st-double cut)

A set of edges E' is a st-double cut, if there is no vertex $v \in V$ that can reach both s and t.

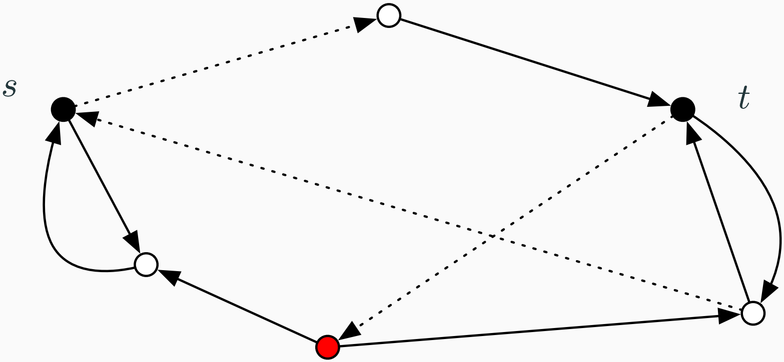
Example



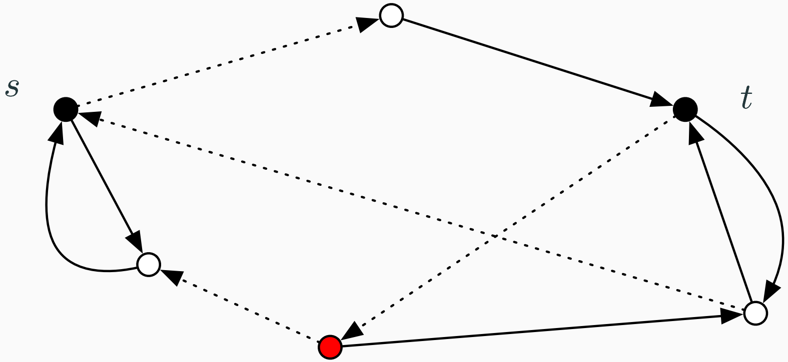
st-bicut example



Not a st-double cut



st-double cut example



Double Cut

st-Double cut

- disconnected s and t: No vertex can reach both s and t.
- $\lambda_d(s, t; G)$ is the value of min st-double cut.
- $\lambda_d(G) = \min_{s,t \in V} \lambda_d(s, t; G)$.

Properties of double cut

Let E' be a st-double cut in G.

- E' is a st-bicut.
- $G - E'$ has no arborescence.

$\lambda_d(G)$ is the minimum number of edges to remove to destroy all arborescence.

Why double cut?

- Blocking arborescence [Bernáth & Pap 2013]
- Application in distributed computing.

Theorem ([Tseng & Vaidya 2015])
The consensus problem in synchronized model can tolerate f edge (node) failure iff remove any f edges (nodes), there is still an arborescence.

- The largest edge failure tolerance is $\lambda_d(G) - 1$.
- The largest node failure tolerance is $\kappa_d(G) - 1$.

How to think about double cut

Theorem ([Bernáth & Pap 2013])
Finding $\lambda_d(s,t)$ is equivalent to finding disjoint sets $S, T \subset V$, such that $s \in S$, $t \in T$ and $d^{in}(S) + d^{in}(T)$ is minimized.

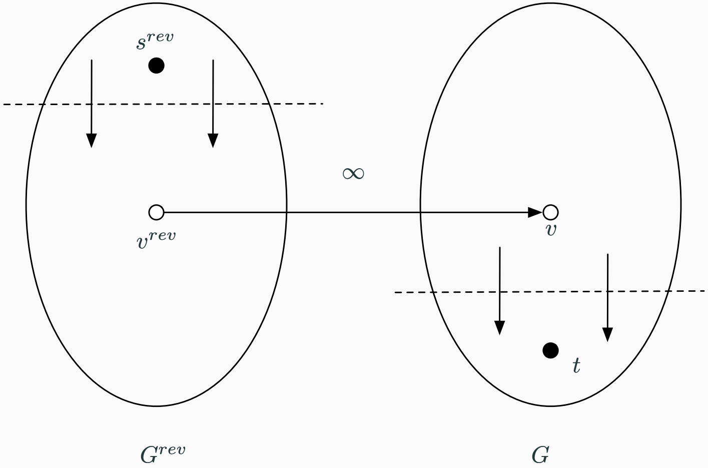
Proof.

- $\Rightarrow$ Let set of vertices that can reach s and t to be S and T, respectively.
- $\Leftarrow$ Remove incoming edges to S and T then no vertices outside S can reach s, outside t can reach T.

$\square$

Corollary
$\lambda_d(s,t)$ can be computed in polynomial time by reducing to maximum flow.

Finding $\lambda_d(s, t)$ through max-flow



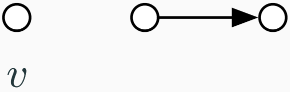
Complexity separation?

There is no complexity separation for local and global double cut.

Node double cut

- $\kappa_d(s, t; G)$ is the value of min st-node double cut.
- $\kappa_d(G) = \min_{s,t \in V} \kappa_d(s, t; G)$.

Difficulty: non-monotonic. A is a node-double cut, $A \cup \{v\}$ might not be a node-double cut.



$A = \emptyset$.

Node Double cut results

Problem	Approximation	Inapproximability
node double cut	2	$3/2-\epsilon$
node *st*-double cut	2	$2-\epsilon$

Open: Is node double cut strictly harder than node *st*-double cut?

Bicut

Bicut

st-bicut: a set of edges such that after its removal there is no path between s and t.

- $\lambda_b(s,t)$ for the value of min *st*-bicut.
- $\lambda_b(G) = \min_{s,t \in V} \lambda_b(s,t)$

st-bicut

A special case of multicut in directed graphs.

- 2-approximation possible. [Dahlhaus et. al. 1994]
- $2 - \epsilon$-inapproximable under UGC. [Chekuri & Madan 16, Lee 16]

How to think about global bicut

A and B are uncomparable if $A \setminus B \neq \emptyset$ and $B \setminus A \neq \emptyset$.

Theorem

The min-bicut problem is equivalent to finding a uncomparable pair $A, B \subset V$ with minimum $|\delta^{in}(A) \cup \delta^{in}(B)|$.

Proof.

- $\Rightarrow$ Remove $\delta^{in}(A) \cup \delta^{in}(B)$, then nodes in $A \setminus B$ cannot reach nodes in $B \setminus A$ and vice versa.
- $\Leftarrow$ no path between s and t. The set of nodes that can reach s and the set of nodes that can reach t are uncomparable, and have in-degree 0.

$\square$

Global bicut

Theorem ([[BCKLX 2017]])
A $(2-\delta)$-approximation, where $\delta \geq \frac{1}{448}$.

A separation between local and global bicut! It's not known if computing $\lambda_b(G)$ is NP-hard.

Results on bicut

A cut is a $s*$-bicut if it is a st-bicut for some t.

Problem	Approximation	Inapproximability
bicut	$2-\delta$	?
$s*$-bicut	2	$4/3-\epsilon$
st-bicut	2	$2-\epsilon$

Node Bicut

st-node-bicut: a set of nodes such that after its removal there is no path between s and t.

- $\kappa_b(s,t)$ for the value of min *st*-bicut.
- $\kappa_b(G) = \min_{s,t \in V} \kappa_b(s,t)$.

min *st*-node bicut and min *st*-bicut are equivalent

- Reduce *st*-bicut to *st*-node bicut: Split each edge by a node, original node have infinite weight.
- Reduce *st*-node bicut to *st*-bicut: Split each node and add edge weight equal to the node weight. All other edges have infinite weight.

The equivalence doesn't hold for global node bicut and global bicut.

Node Bicut results

Problem	Approximation	Inapproximability
node bicut	2	$3/2 - \epsilon$
node $s*$-bicut	2	$3/2 - \epsilon$
node st-bicut	2	$2 - \epsilon$

Open: Is there a complexity separation between node bicut and st-node bicut?

Disconnecting more than 2 vertices

k-cuts in undirected graphs

A set $T \subset V$ is disconnected if there is no path between any pair of vertices in T.

- k-cut: A set of edges s.t. its removal creates a disconnected set of size at least k.
- T-separating k-cut: A k-cut s.t. its removal disconnects T.
- $\lambda^k(T; G)$ is the value of min T-separating k-cut.
- Local: $\lambda^k(T; G)$ for $|T| = k$.
- Global: $\lambda^k(G)$, is the value of min k-cut.

$$\lambda^k(G) = \min_{|T|=k, T\subset V} \lambda^k(T; G).$$

  $\lambda^k(G) = \lambda^k(T, G)$ where $|T| \leq 1$.
- Semiglobal: $|T| = i$ for some $2 \leq i < k$.

Previous results

Let k be a constant.

- Computing $\lambda^k(G)$ can be done in polynomial time. [Goldschmidt & Hochbaum 1994]
- Computing $\lambda^k(T; G)$ is hard for $|T| \geq 3$, but admits a 2-approximation. [Garg, Vazirani & Yannakakis 2004] In particular, min local k-cut for $k \geq 3$ is NP-hard.

Local k-cut is strictly harder than the global k-cut.

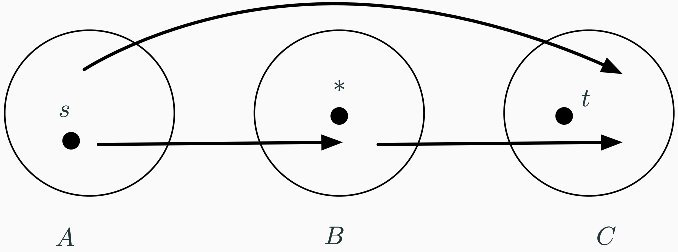
A subproblem for bicut approximation: $s * t$-linear 3-cut

A set of edges E' is $s * t$-linear 3-cut if there exist a vertex $r \neq s, t$, such that s cannot reach r and t, and r cannot reach t in $G - E'$.

Theorem

Finding $s * t$-linear 3-cut is equivalent to finding

$$\min\left\{d(A,B)+d(A,C)+d(B,C) : {\{A,B,C\} \text{ a partition of } V, \atop A,B,C\neq\emptyset, s\in A, t\in C}\right\}.$$



Why do we care about $s * t$ linear 3-cut

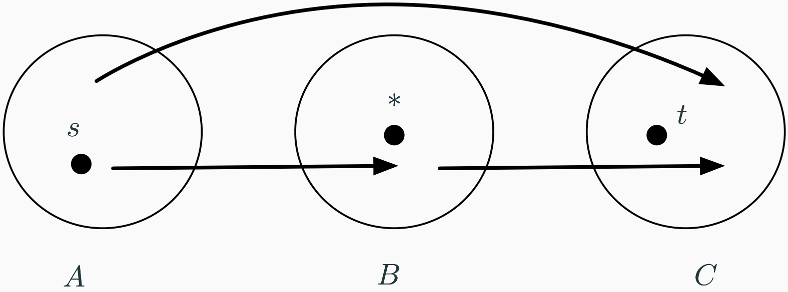
- It's a semiglobal version of linear k-cut. [Chekuri & Madan 2017]
- Improvement in approximation of $s * t$ linear 3-cut improves the $2 - \delta$ approximation algorithm for min bicut.

- Finding a minimum $s * t$-linear 3-cut is not known to be NP-hard.
- We've shown there is a $3/2$-approximation algorithm.
- A newer result shows there is a $\sqrt{2}$-approximation algorithm. [Bérczi et. al. unpublished]

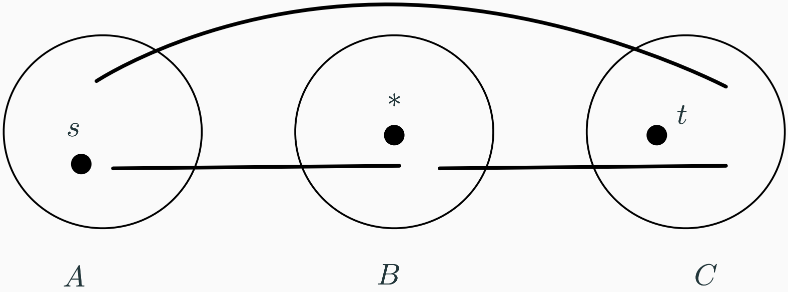
Is $s * t$-linear 3-cut NP-hard?

What about the undirected version?

What is the undirected version of $s * t$-linear 3-cut?



What is the undirected version of $s * t$-linear 3-cut?



Undirected version of $s * t$-linear 3-cut is $\{s, t\}$-separating 3-cut.

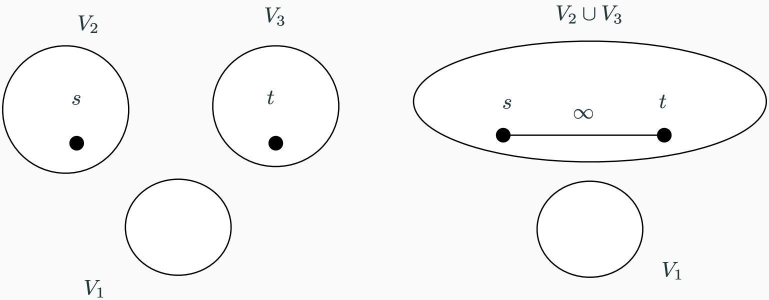
$\{s,t\}$-separating edge k-cut

We want to find $\lambda^k(T;G)$ for $|T| = 2$.

1. At the boundary between P and NP-hard.
    - Finding $\lambda^k(T,G)$ is NP-hard for $|T| \geq 3$.
    - Finding $\lambda^k(T,G)$ is easy for $|T| \leq 1$.
2. An open problem [Queyranne 2012].

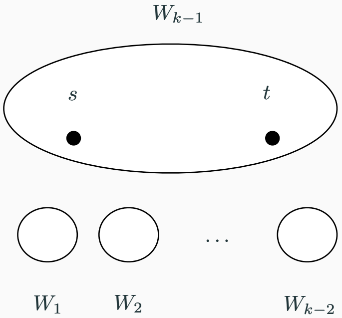
Theorem ([BCKLX 2017])
Let $\{V_1, \ldots, V_k\}$ be a partition of V corresponding to an optimal solution of min st-separating k-cut in G. $s \in V_{k-1}$ and $t \in V_k$. Add a infinite weight edge between st and call the new graph H.

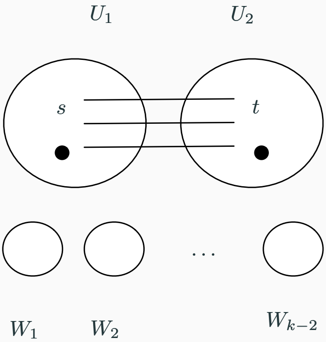
$$c(V_1, \ldots, V_{k-2}, V_{k-1} \cup V_k) \leq 2\lambda^{k-1}(H)$$



Proof

Let $W_1, \ldots, W_{k-1}$ be a optimal $k-1$ cut for H and $s, t \in W_{k-1}$.
Let U_1 and U_2 be min st-cut in $G[W_{k-1}]$
$\lambda^k(G) \leq c(W_1, \ldots, W_{k-2}, U_1, U_2) = \lambda^{k-1}(H) + \lambda(s, t; G[W_{k-1}])$.

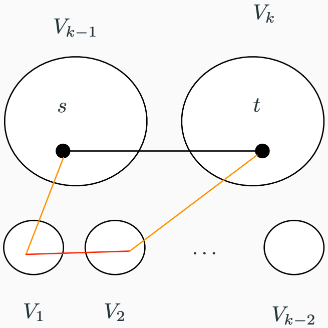




Proof

Let $V_1, \ldots, V_k$ be a optimal k-cut for G, $s \in V_{k-1}$, $t \in V_k$.

$$\lambda(s,t;G) \leq d(V_{k-1}, V_k) + \frac{1}{2}\left(d(V_1 \cup \ldots \cup V_{k-2}) + \sum_{i,j \leq k-2, i \neq j} d(V_i, V_j) \right)$$



$d(V_{k-1}, V_k)$

$d(V_1 \cup \ldots \cup V_{k-2})$

$\sum_{i,j \leq k-2, i \neq j} d(V_i, V_j)$

Proof

$$
\begin{aligned}
&\lambda^{k-1}(H) + \lambda(s,t;G[W_{k-1}]) \\
&\geq \lambda^{k}(G) \\
&= d(V_{k-1},V_k) + d(V_1 \cup \ldots \cup V_{k-2}) + \sum_{i,j\leq k-2, i\neq j} d(V_i,V_j) \\
&\geq \lambda(s,t;G) + \frac{1}{2}\left(d(V_1 \cup \ldots \cup V_{k-2}) + \sum_{i,j\leq k-2, i\neq j} d(V_i,V_j) \right) \\
&= \lambda(s,t;G) + \frac{1}{2}(c(V_1,\ldots,V_{k-2},V_{k-1}\cup V_k)) \\
&\geq \lambda(s,t;G[W_{k-1}]) + \frac{1}{2}(c(V_1,\ldots,V_{k-2},V_{k-1}\cup V_k))
\end{aligned}
$$

Algorithm for st-separating k-cut

Algorithm

1. Enumerate all $k-1$-cut $\{W_1, \ldots, W_{k-1}\}$ with value at most $2\lambda^{k-1}(H)$, assuming $s, t \in W_{k-1}$.
2. For each $k-1$-cut, find min-st-cut in $G[W_{k-1}]$, say $\{U_1, U_2\}$. Let $\{W_1, \ldots, W_{k-2}, U_1, U_2\}$ be a candidate solution.
3. Output the candidate solution with the smallest value.

There are $O(n^{2(k-1)})$ $k-1$-partitions with value $\leq 2\lambda^{k-1}(H)$. [Karger & Stein 1996]

Theorem ([BCKLX 2017])
The st-separating k-cut can be solved in polynomial time for constant k.

Node k-cuts

1. $\kappa^k(T; G)$ is the value of the minimum T-separating node k-cut.
2. $\kappa^k(G)$ is the value of the minimum node k-cut.

Previous Results

1. $\kappa^k(T; G)$ has a $(2 - 2/k)$-approximation [Garg, Vazirani & Yannakakis 2004].
2. It was raised as an open problem if $\kappa^k(G)$ is solvable in polynomial time for all $k \geq 3$. [Goldschmidt & Hochbaum 1994]

Node k-cut results

A complete characterization for node-k-cut.

Theorem ([BCKLX 2017])
If $k \geq 3$, then there exist a $(2 - 2/k)$-approximation algorithm for $\kappa^k(T; G)$ and cannot be approximated within $(2 - 2/k - \epsilon)$. Otherwise, it's polynomial time solvable.

(H, t)-cuts

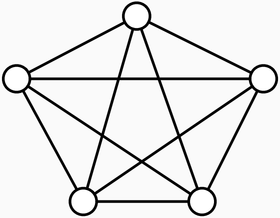
(H,t)-cuts

- H a graph(digraph) on $\{1,\ldots,k\}$, and a integer t. H is called the pattern graph.
- $G=(V,E)$ be a input graph(digraph)
- A k-partition $(V_1,\ldots,V_k)$ of V where $V_{t+1},\ldots,V_k$ are non-empty is a (H,t)-cut. ($V_1,\ldots,V_t$ can be empty)
- The (H,t)-cut value of $(V_1,\ldots,V_k)$ is

$$\sum_{\substack{e\in V_i\to V_j\\(i,j)\in E(H)}} w(e)$$

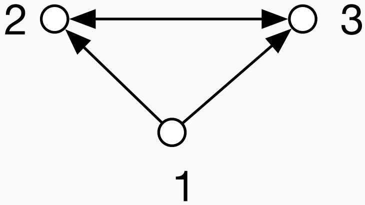
- What can we model with (H,t)-cut?

k-cut



$t = 0$

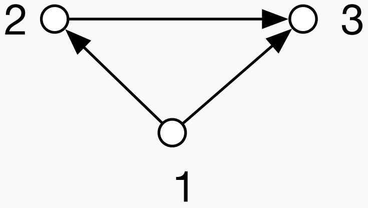
Double cut



$t = 1$.

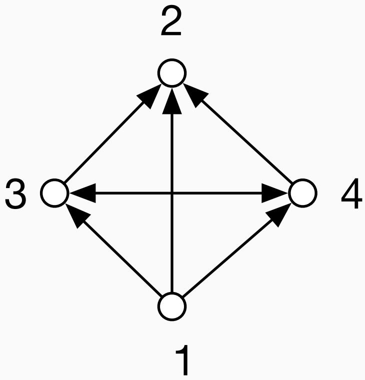
Find A and B such that $A \cap B = \emptyset$ and $|\delta^{in}(A)| + |\delta^{in}(B)|$ is minimized.

Linear 3-cut



$t = 0.$

bicut



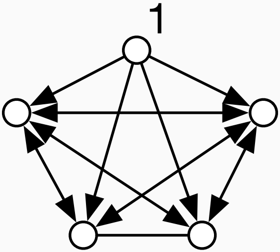
$t = 2$. Find two uncomparable sets A and B such that $|\delta^{in}(A) \cup \delta^{in}(B)|$ is minimized. Let $V_1 = V \setminus (A \cup B)$, $V_2 = A \cap B$, $V_3 = A \setminus B$, $V_4 = B \setminus A$.

k-subpartition

Find k sets $\{V_1, \ldots, V_k\}$ such that $V_i \cap V_j = \emptyset$ and minimize

$$\sum_{i=1}^{k} |\delta^{in}(V_i)|.$$

Double cut is equivalent to 2-subpartition.



$t = 1$. Solvable in polynomial time if G is obtained from bidirect all edges of a undirected graph [Nagamochi 2007].

Polynomial time solvable cases when H is undirected

- If H has at most 4 vertices, then finding min $(H, 0)$-cut is NP-hard iff $H = 2K_2$. [Elem, Hassin & Monnot 2013 unpublished] Reduces to partition the graph to two disjoint bicliques.

A vertex v is neighborhood minimal, if there is no vertex u such that $N(u) \subsetneq N(v)$. min-$(H, 0)$-cut is solvable in polynomial time if

- The neighborhood minimal vertices of H is a independent set in H.
- $H = H_1 + H_2$ where min-$(H_1, 0)$-cut and min-$(H_2, 0)$-cut are solvable in polynomial time. [Kawarabayashi and X unpublished]

Fixing terminals

Given G and $U_1, \ldots, U_k$, find min-(H, t)-cut $(V_1, \ldots, V_k)$ such that $U_i \subset V_i$.

Open Problems

Polynomial time algorithms for (H, t)-cut

- For which (H, t) pair is min (H, t)-cut solvable in polynomial time?
- Does $(H, 0)$-cut solvable in polynomial time implies (H, t)-cut solvable in polynomial time for all t?
- What about fixed terminal version?

Close the gaps

Problem	Edge-deletion	Node-deletion
DoubleCut	Poly-time	2-approx $(3/2 - \epsilon)$-inapprox
st-DoubleCut	Poly-time	2-approx $(2 - \epsilon)$-inapprox
BiCut	$(2 - 1/448)$-approx NP-hard?	2-approx $(3/2 - \epsilon)$-inapprox
$s*$-BiCut	2-approx $(4/3 - \epsilon)$-inapprox	2-approx $(3/2 - \epsilon)$-inapprox
st-BiCut	2-approx $(2 - \epsilon)$-inapprox	[Equivalent to edge-deletion]
$s * t$-Linear 3-cut	$\sqrt{2}$-approx NP-hard?	2-approx $(4/3 - \epsilon)$-inapprox

Hypergraphs

$\lambda^k(G)$ can be found in hypergraphs in randomized polynomial time [Chandrasekaran, X & Yu unpublished].

What about $\lambda^k(\{s,t\}\,;G)$?

- The algorithm is still correct.
- Number of approximate min-k-cut is exponential.
- Exponential running time.

Can we find $\lambda^k(\{s,t\}\,;G)$ in polynomial time for hypergraphs?

Thank You!