Computing min-cuts in hypergraphs

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A hypergraph

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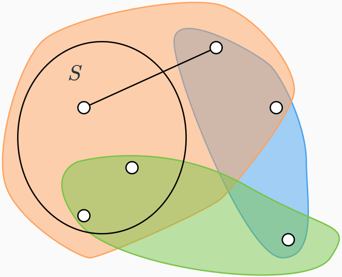
Each edge is assigned with a positive weight $w : E \to \mathbb{R}_+$.

Cut function

- Edge $e \in E$ crosses $S \subset V$, if $e \cap S$ and $e \cap V \setminus S$ are both non-empty.
- $\delta(S)$ is the set of all edges cross S.
- The cut function $c : 2^V \to \mathbb{R}_+$

$$c(S) = \sum_{e \in \delta(S)} w(e)$$



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- A cut S a *st*-cut, if $s \in S$ and $t \in V \setminus S$.
- The max-flow min-cut theorem is relates max-*st*-flow with min-*st*-cut (also called local min-cuts). It's value is denoted as $\lambda(G, s, t)$.

Why min-cut?

The value of min-cut measures the connectivity of the graph.

Hence min-cuts are useful in clustering and combinatorial optimization.

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Our contribution: a deterministic algorithm that finds ALL min-cut in a hypergraph in the same running time as finding a *single* min-cut.

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- How to find a single min-cut in a graph?
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- A new algorithm for finding ALL min-cuts in graphs.
- How to modify the algorithm to work for hypergraphs?

n is the # of vertices. m is the # of edges.

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  $n - 1$ max flows $= O(n^2 m)$

- Fastest known:

$$\lambda(G) = \min(\lambda(G, s, t), \lambda(G/st))$$

  $O(nm + n^2 \log n)$ [Nagamochi & Ibaraki 1992]

The min-cut algorithm

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- $n - 1$ max flow computation.
- $O(n^2 m)$ again.
- How to improve this?

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- How? maximum adjacency ordering.

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$v_1, \ldots, v_n$ is a maximum adjacency ordering(MA-ordering) if for all $1 \leq i \leq j \leq n$,

$$d(\{v_1, \ldots, v_{i-1}\}, v_i) \geq d(\{v_1, \ldots, v_{i-1}\}, v_j).$$

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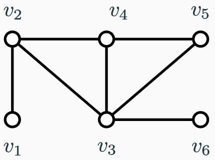

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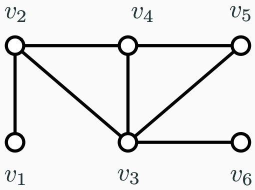

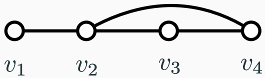

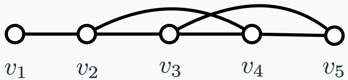

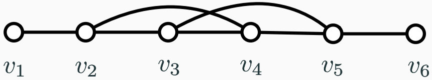

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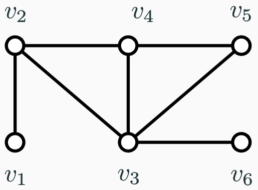

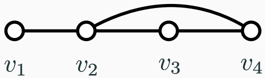

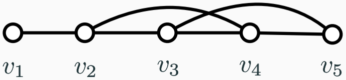

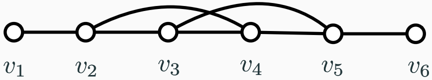

MA-ordering can be found in $O(m + n \log n)$ time.

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Lemma ([Arikati & Mehlhorn 1999])

Given graph $G = (V, E)$ and MA ordering $v_1, \ldots, v_n$. A max $v_{n-1}v_n$-flow can be found in $O(m)$ time.

A min-cut of a graph can be found in $n - 1 \times$ MA-ordering, so $O(nm + n^2 \log n)$ time.

What about hypergraphs?

- The exact same algorithm works for hypergraphs.
- Different ordering:
  - MA ordering [Klimmek & Wagner 1996]
  - Tight ordering [Mak & Wong 2000]
  - Queyranne ordering [Queyranne 1998]

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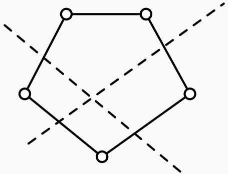
- $O(n^3)$ space sufficient: $(S, V \setminus S)$ requires $\min(|S|, |V \setminus S|)$ space.
- $\Omega(n^3)$ space required: a cycle, space usage $\sum_{i=1}^{n/2} ni = \Omega(n^3)$.



Finding ALL min-cuts: desirable properties

Find a data structure:

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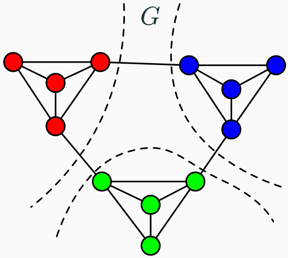
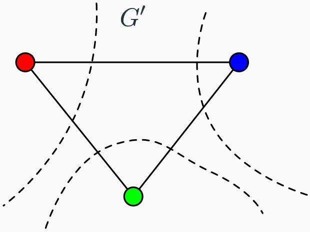
The best kind of data structure: a smaller, simple graph with the same min-cut structure.

Representation

$G' = (V', E')$ is a representation of $G = (V, E)$ if there exist a function $\phi : V \to V'$, such that

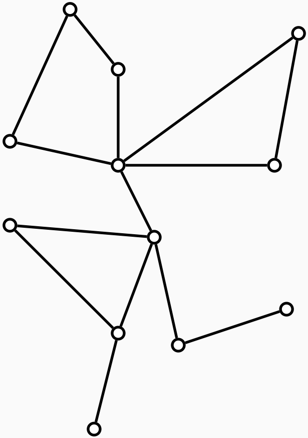
- S is a min-cut in G, then $\phi(S)$ is a min-cut in G'.
- S' is a min-cut in G', then $\phi^{-1}(S')$ is a min-cut in G.





Cactus

A graph is called a cactus if no two cycles share an edge.



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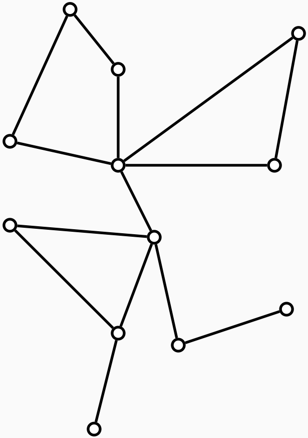


Every graph has a cactus representation. [Karzanov & Timofeev 1986]

Cactus representation: Algorithms, a long history

Large number of work

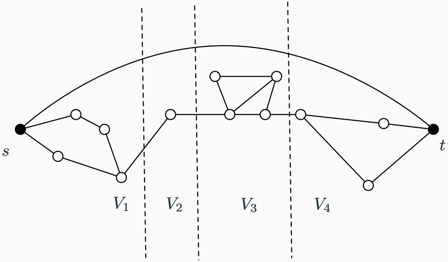
- $O(mn\log(n^2/m))$ [Gabow 1993]
- $O(mn + n^2\log n + n^* m\log n)$ [Nagamochi & Kameda 1996]
  (n^* is # of vertices in the representation)
- $O(mn + n^2\log n + \gamma m\log n)$ [Nagamochi, Nakao, Ibaraki 2000]
  (γ is # of cycles in the representation)
- $O(mn + n^2\log n)$ [Nagamochi, Nakamura, Ishii 2003]

Cactus representation: The state of the art

An edge e is critical, if e crosses a min-cut.

Theorem ([Nagamochi & Kameda 1996])

If st is an critical edge, then there exists a partition $\{V_1, \ldots, V_k\}$ of V, such that $\left\{\bigcup_{i=1}^{j} V_i | 1 \leq j < k\right\}$ are the set of min-st-cuts.



Can be found in $O(m + n \log n)$ time: MA-ordering.

Cactus representation: Previous method

The partition give us a partial cactus.

Algorithm $\textsc{All-MinCut}(\mathsf{G})$:

1. $A \leftarrow$ cactus of all min-st-cut for some critical edge st.
2. $B \leftarrow \textsc{All-MinCut}(G/st)$.
3. Cleverly combine cactus A and B into a cactus for G.

Running time$=n-1\times$ MA ordering, same as finding a single min-cut.

Why is the algorithm unsatisfactory

- Complicated

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- Complicated
- Does not generalize to hypergraphs.

Finding cactus representation for graphs

Our approach: a new algorithm

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Remark: Decomposition framework actually handles all (symmetric) submodular functions ([Fujishige 1983]) [Cunningham 1983].

Decomposition: Refinement

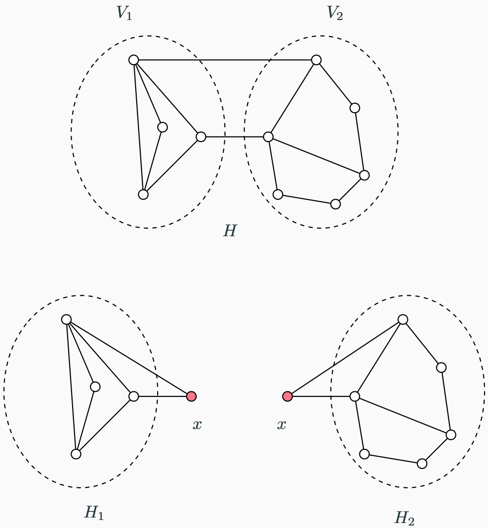
A min-cut with at least two vertices on each side is called a split.

Definition

Given a graph G, $\{G_1, G_2\}$ is a refinement of G if G_1 and G_2 are graphs obtained through a split (V_1, V_2) and a new marker vertex x as follows.

1. G_1 is G/V_2, such that V_2 gets contracted to x.
2. G_2 is G/V_1, such that V_1 gets contracted to x.

Example of a refinement



Decomposition

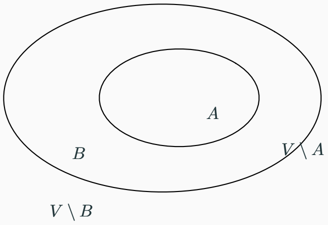
Definition

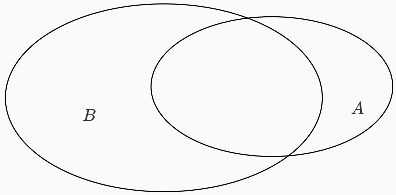
A set of graphs $\mathcal{D} = \{G_1, \ldots, G_k\}$ is a decomposition of G if it is obtained from $\{G\}$ by replacing an element by its refinements.

Two decomposition are equivalent if they are the same up to relabeling marker vertices.

Crossing

Two cuts $(A, V \setminus A)$ and $(B, V \setminus B)$ are crossing if $A \cap B$, $A \setminus B$ and $B \setminus A$ are all non-empty.

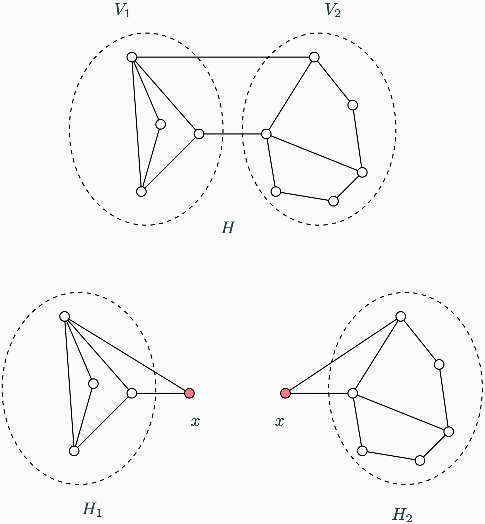




If $(A, V \setminus A)$ and $(B, V \setminus B)$ are non-crossing, then contract A or B preserves the other cut.

A split is good if no min-cut crosses it.

Refinement preserves cuts



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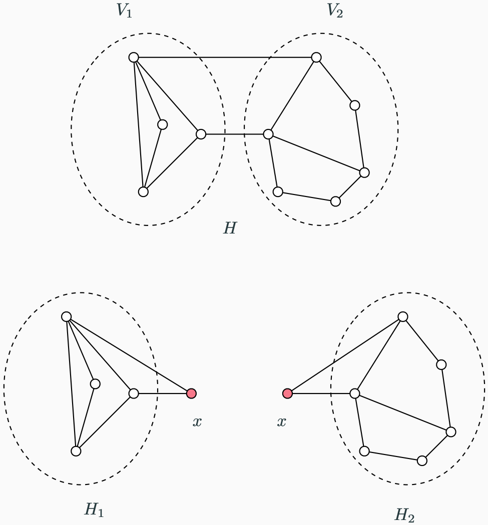
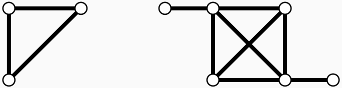


The min-cut in H_1 and H_2 are the min-cuts in H.

Refine through good splits preserves all min-cuts.

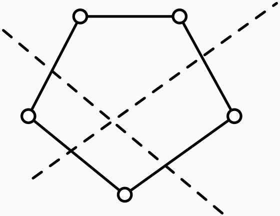
Definition

A graph G is prime if it does not contain any split.



Good splits and cycles

There exist graphs that are not prime but have no good splits.



Theorem

A graph without any good split is either a prime or a cycle.

Standard decomposition

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A standard decomposition is minimal, if it is not a refinement of any other standard decomposition.

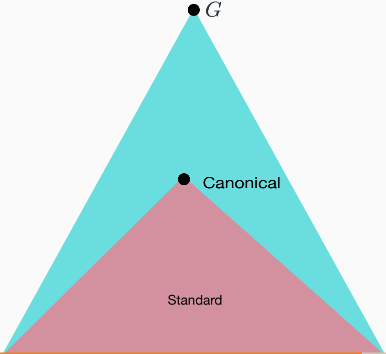
Canonical decomposition

Theorem ([Fujishige 1983,Cunningham 1983])

There exist a unique minimal standard decomposition.

Such decomposition is called the canonical decomposition.

Canonical decomposition is what we want: obtained through only applying good splits.



Properties of the canonical decomposition

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New Goal: Find the canonical decomposition.

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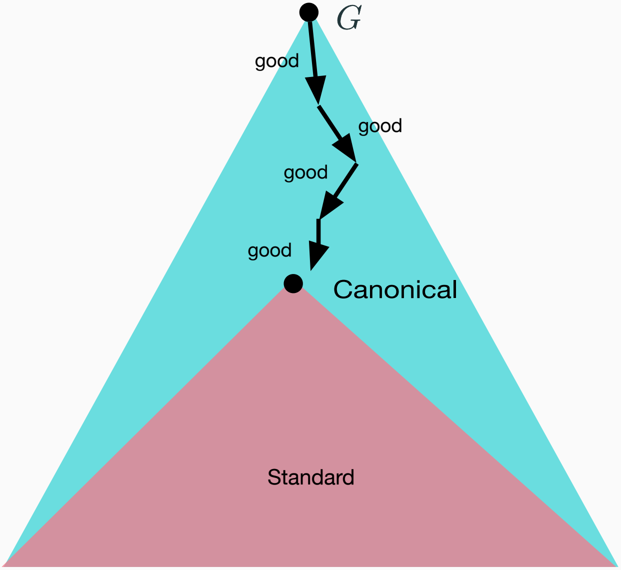
Algorithm running time are dominated by finding $O(n)$ good splits.
Finding a good split is no easier than finding a min-cut.


G
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Canonical
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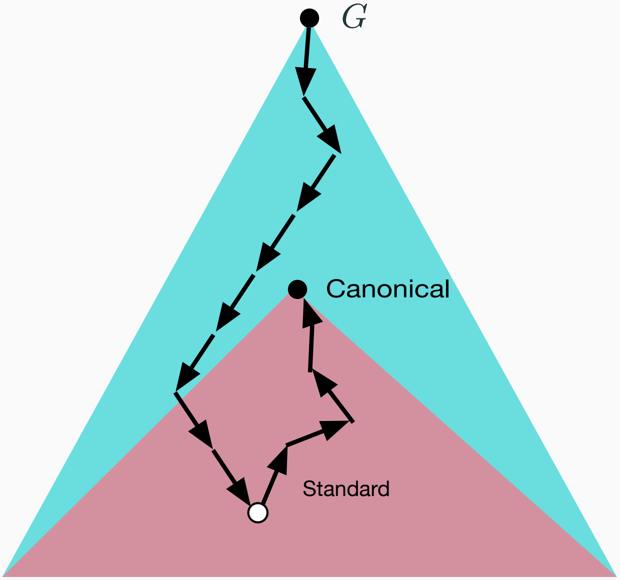
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Algorithm, second attempt

Merge: $\{G_1, \ldots, G_k\}$ a decomposition of G, G_1 and G_2 shares a marker vertex. Find a decomposition $\{G', G_3, \ldots, G_k\}$ of G, where $\{G_1, G_2\}$ is a refinement of G'.

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How to find a standard decomposition

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For fixed $s, t \in V$, either there is a split that separates s and t(called a *st*-split). Or we can contract s and t.

Split oracle

Problem

Given G and the min-cut value λ, outputs either a split in G or a pair of vertices $\{s, t\}$ such that there is no st-split in G.

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If we have a fast split oracle, then we can solve the problem.

1. Apply the split oracle on G.
2. If the oracle returns a split, find a refinement of H and recurse on both sides.
3. Otherwise, the oracle output a pair s, t. Contract s and t and recurse.

Theorem

A split oracle can be implemented in $O(m + n \log n)$ time.

Sketch

- Use MA-ordering to find a max *st*-flow for some s and t.
- Enumerate at most 3 min *st*-cuts using the maximum flow. It either finds a split or decide there is none. [Provan & Shier 1996]

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Total running time: $O(nm + n^2 \log n)$. Same as finding a single min-cut.

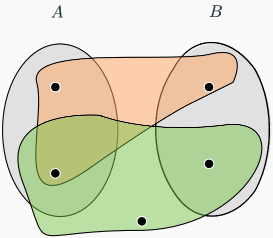
What about hypergraphs?

Our algorithm almost work for hypergraphs. Just a few small changes.

1. We need to use tight ordering instead of MA-ordering.
2. There are non-cycles that also doesn't have any good split.
3. Cactus is not the right representation.

Modification 1: Tight adjacency and tight ordering

$$d'(A, B) = \sum_{\substack{e \in \delta(A) \cap \delta(B) \\ e \subseteq A \cup B}} w(e)$$



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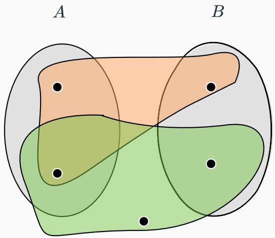

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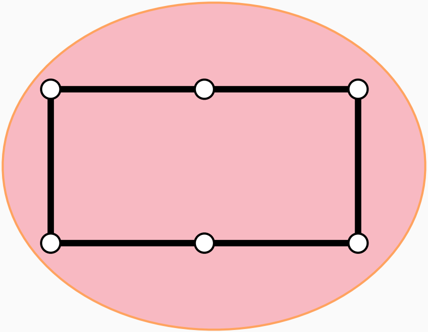
$v_1, \ldots, v_n$ is a tight ordering(MA ordering) if for all $1 \leq i \leq j \leq n$,

$$d'(\{v_1, \ldots, v_{i-1}\}, v_i) \geq d'(\{v_1, \ldots, v_{i-1}\}, v_j).$$

Modification 2: Solid polygons



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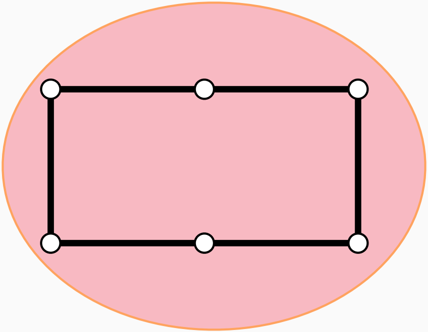

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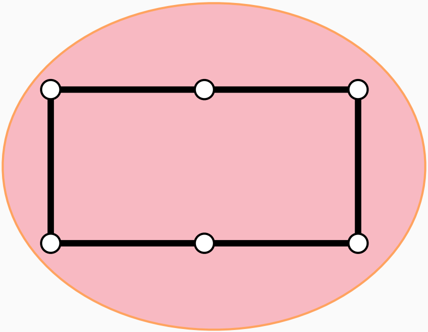
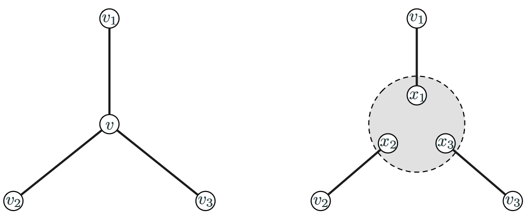


A hypergraph is a solid polygon if it consist of a (possibly 0 weight) cycle and a (possibly 0 weight) hyperedge covering all vertices. Hypergraphs with no good splits are either primes or solid polygons. A standard decomposition consist of primes and solid polygons.

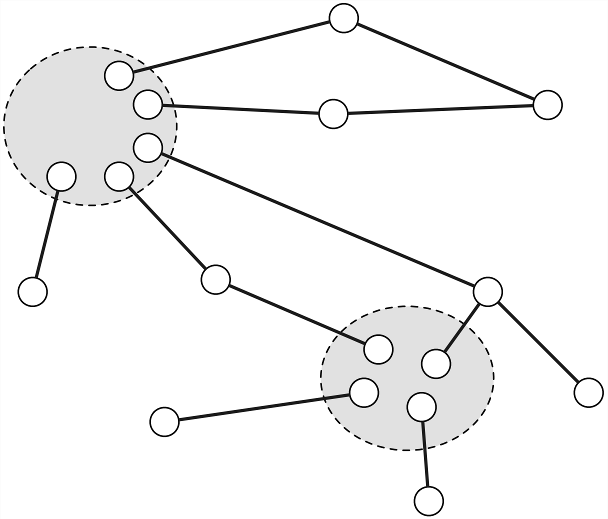
Modification 3: Hypercactus

Definition

A hypergraph is called a hypercactus if it can be obtained through the following operations from a cactus: Split the edges incident to a star, and "blow up" the vertex into a hyperedge.



Modification 3: Hypercactus Example



Modification 3: Hypercactus representation

Theorem ([Cheng 1999, Fleiner & Jordán 1999])
Every hypergraph have a hypercactus representation.

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Remark: Fleiner & Jordán actually showed there is a hypercactus representation for all symmetric submodular functions.

Conclusion

All hypergraph min-cuts can be computed in the same running time as finding a single min-cut.

Thank you!