Minimum violation maps and their applications to cut problems

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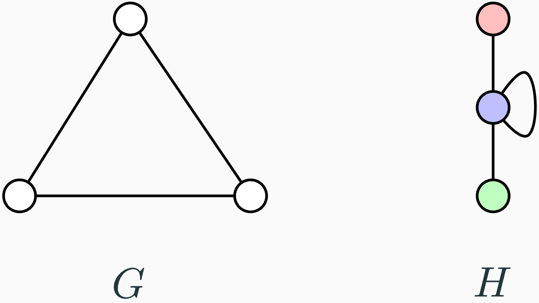
Map that preserve graph structure

A map from $G = (V, E)$ to $H = (U, F)$ is a function $f : V \to U$. H is the pattern graph.

A map is a (graph) homomorphism, if $uv \in E$ then $f(u)f(v) \in F$.



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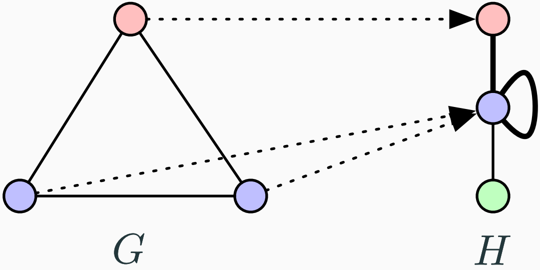

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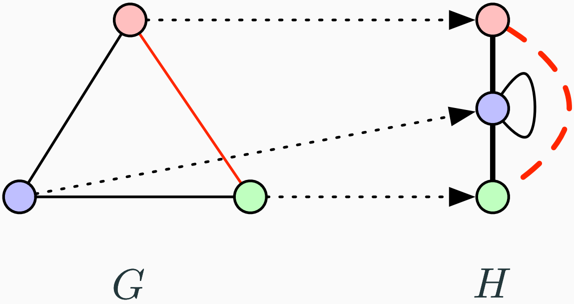

Graph homomorphism model other problems

Fix a pattern graph, is the graph homomorphism problem hard?

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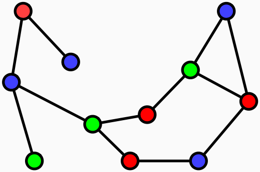
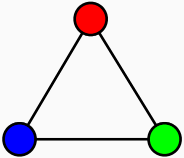
Is the graph 3-colorable?





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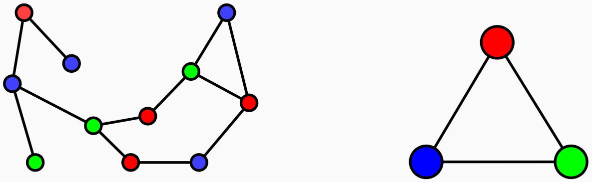

Graph homomorphism is NP-hard.

Graph homomorphism can be easy (even with constratints)

The graph homomorphism problem for any graph with a self-loop is trivial: map all vertices to a vertex with self loop.

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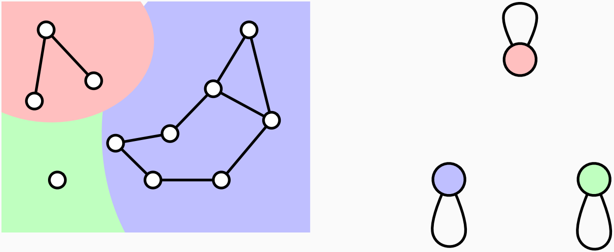
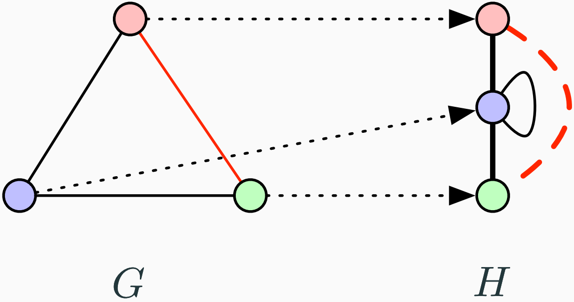
Does the graph have 3 components?



* additional surjectivity is required.

Measure how far away from homomorphism

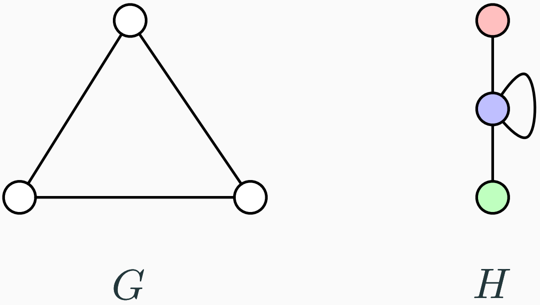
The edge not mapped to an edge is a violating edge.



The violation of a map is the number of violating edges.

Surjective Minimum Violation. SVio(H)
Input: $G = (V, E)$.

Output: A surjective map from G to H with minimum violation.



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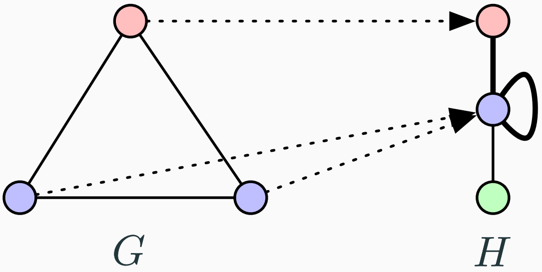

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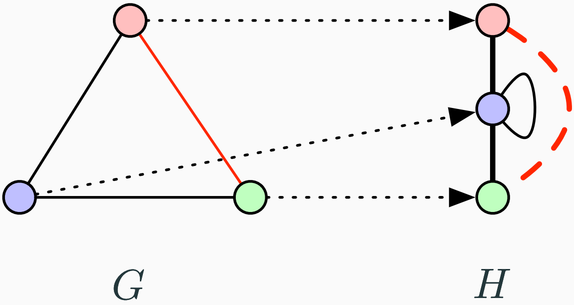

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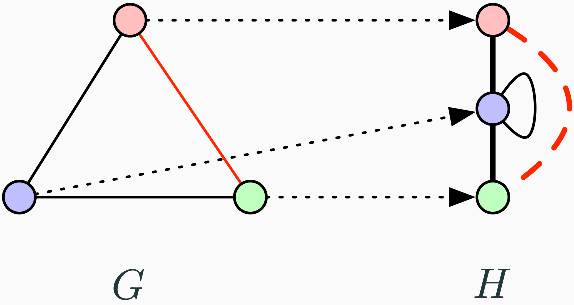


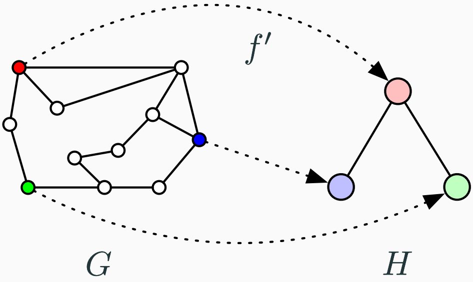
H is s-tractable if SVio(H) is tractable.

Retraction minimum violation. RVIO(H)

Input: graph G and a bijection $f' : V' \to U$ for some $V' \subseteq V(G)$

Output: A map f from G to H such that $f|_{V'} = f'$ and the violation is minimized.

Vertices in V' are called terminals.



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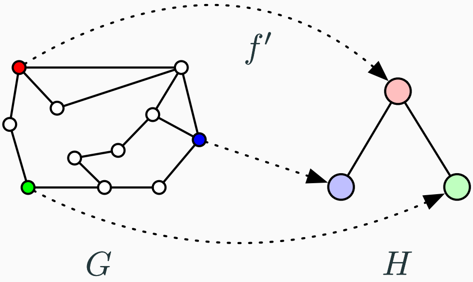

H is r-tractable if RVio(H) is tractable.

Goal: classify the s-tractable and r-tractable graphs.

Why?

For a cut problem, there is usually a pattern graph H, such that

Fixed-Terminal Min Cut $\in$ **P**	$\Leftrightarrow$	H is r-tractable
Global Min Cut $\in$ **P**	$\Leftrightarrow$	H is s-tractable

Consequence: A unified tool to quickly decide if a cut problem is easy or hard by looking at the pattern graph H.

Out Results

- Relating r-tractability and s-tractability with various cut problems.
- A **complete classification** of r-tractable graphs.
- disconnected reflexive s-tractable graphs are defined by the s-tractability of its components.
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Remarks:

- All graphs after this point are reflexive. For simplicity, we do not draw the self-loops.
- We state theorems for graphs, but there are directed graph counterparts.
- Our results hold for weighted graphs too. The violation is the sum of the weights of the violating edges.

Previous Work

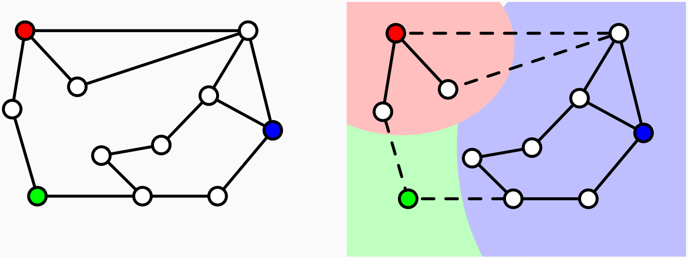
- Classification of s-tractable graphs and r-tractable graphs was studied under the name "G_c-cut". [Elem, Hassin & Monnot 13]
- A more general problem called 0-extension problem was studied, but only approximation was considered [Calinescu et. al. 01]

Modeling cut problems

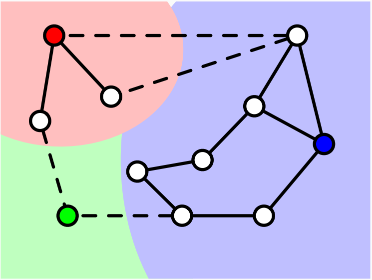
Problem: Min k-way cut

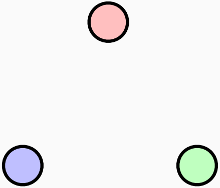
Input: G and the terminals $v_1, \ldots, v_k \in V(G)$

Output: Find a minimum set of edges F, such that such in $G - F$, each pair of terminals cannot reach each other.



3-way cut





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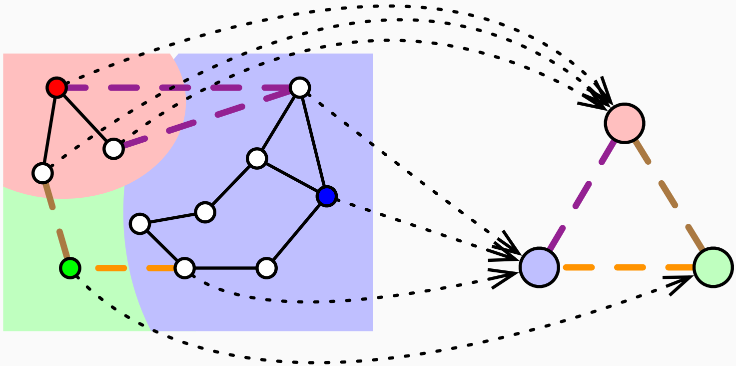

k-cut

Problem: min k-cut

Input: $G = (V, E)$

Output: A minimum set of edges F, such that $G - F$ has at least k components.

k-way cut and k-cut

I_k is the reflexive graph of k isolated vertices.

k-way cut is equivalent to $\textsc{RVio}(I_k)$.

k-way cut is NP-hard for $k \geq 3$. [Dahlhaus et. al. 94]

k-cut is equivalent to $\textsc{SVio}(I_k)$.

Solvable in polynomial time for every fixed k [Goldschmidt & Hochbaum 94, Karger & Stein 96].

Theorem
I_k is r-tractable if and only if $k \leq 2$. I_k is s-tractable for all k.

(ℓ, k)-way-cut

A set of edges F is a (ℓ, k)-way-cut for a set of terminals $v_1, \ldots, v_k$, if in $G - F$, the pairwise distance of the terminals is at least $\ell + 1$.

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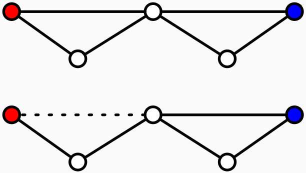
Problem: (ℓ, k)-way-cut
Input: G, terminals $v_1, \ldots, v_k$

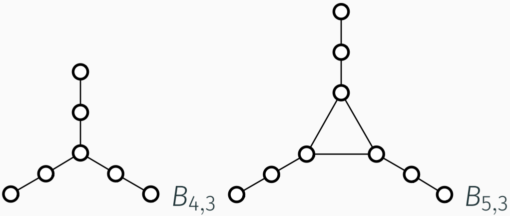
Output: A minimum cardinality (ℓ, k)-way-cut.



$(2, 2)$-cut

Theorem ([Mahjoub & McCormick 00])
$(\ell, 2)$-way-cut is tractable if and only if $\ell \leq 3$.

Theorem
(ℓ, k)-way-cut is equivalent to **RVIO**$(B_{\ell,k})$.



Cut problem in directed graphs

F is a set of edges. F is a

- k-reach-cut, if in $G - F$, there exists a set of k terminals, such that every vertex can reach at most one of the terminals.

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- Bicut, if in $G - F$, there are two vertices s and t that cannot reach each other.

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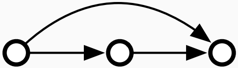
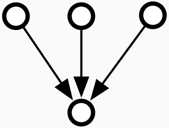
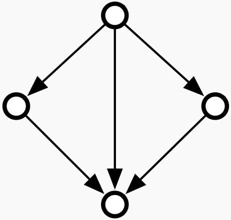


(a) T_3, lineat-3-cut



(b) S_k, $k = 3$, 3-reach-cut



(c) H_{bicut}, bicut

Previous cut problem is equivalent to **SVIO**(H) for some directed graph H.

Classification of r-tractable graphs

Start with a harder problem.

Cost

Let $G = (V, E)$, $H = (U, F)$. A cost function $c : V \times U \to \mathbb{N}$ assigns cost $c(v, u)$ to mapping v to u.

The cost of a map f from G to H is

$$\sum_{v \in V} c(v, f(v))$$

Minimum cost and violation

Problem: CVIO(H)

Input: Graph G and a cost function c.

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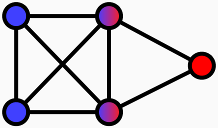
H is c-tractable if CVio(H) is tractable.

Theorem ([Deineko et.al. 08])
H is c-tractable if and only if there are two sets A and B such that $A \cup B = V$, and $H[A]$ and $H[B]$ are cliques.



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Input of RVɪᴏ(H) is G and $f' : V' \to U$.

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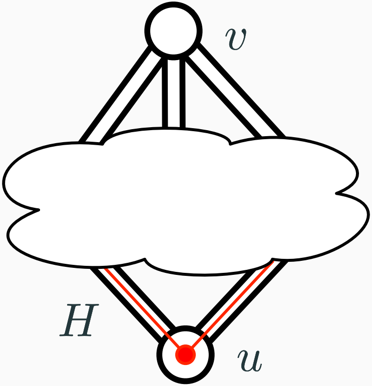
Nope: P_4 is r-tractable but c-tractable.

An observation: moving up



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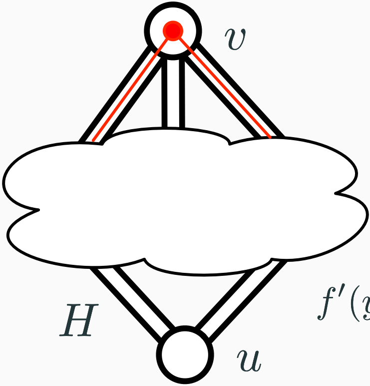


$$N(u) \subseteq N(v)$$
$$f(x) = u$$

$$f'(y) = \begin{cases} f(y) & \text{if } y \neq x \\ v & \text{otherwise} \end{cases}$$

Violation of f' is at most violation of f.

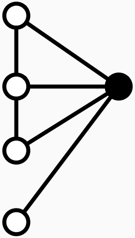
Superseded vertices

Assume there is a total order $\prec$ of the vertices in H. u is superseded by v, if

- $N(u) \subsetneq N(v)$, or
- $N(u) = N(v)$ and $u \prec v$.

Apex

A vertex is an apex if no vertex supersedes it.



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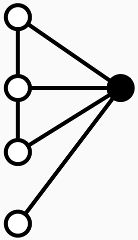
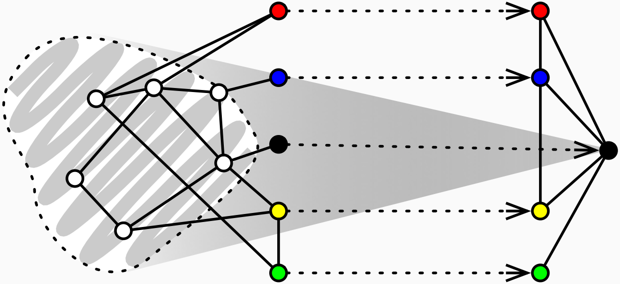
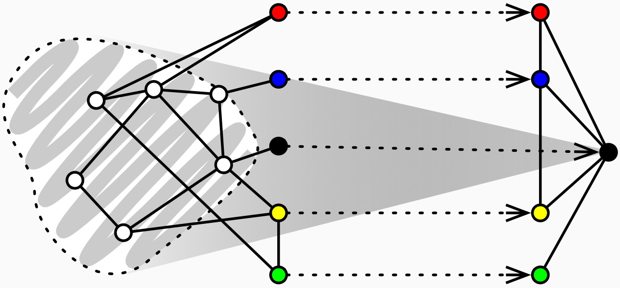


The apex subgraph of H is $H[A]$, where A is the set of apex vertices.

There exists an optimal solution where the non-fixed vertices are mapped to apex vertices.





A graph with a single vertex apex subgraph is r-tractable.

Theorem ([Elem, Hassin & Monnot 13])
H is r-tractable if the apex subgraph of H is a complete graph.

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Complete graphs are c-tractable.

r-tractability and c-tractability

Theorem ([Kawarabayashi & X 20])
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$\mathbf{RVio}(H)$

$G = (V, E)$

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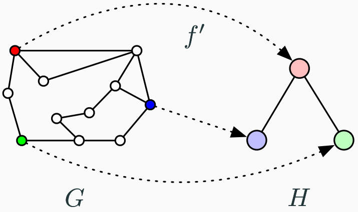
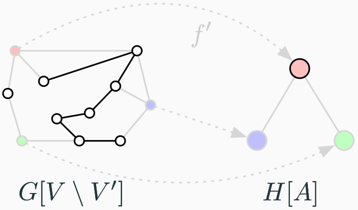


$\mathbf{CVio}(H[A])$

$G[V \setminus V']$

$$c(v,u) = \left| \left\{ vv' \;\middle|\; \begin{array}{l} v' \in V' \\ vv' \in E \\ uf'(v') \notin F \end{array} \right\} \right|$$



Classification of RVIO

The harder direction requires knowledge from CSP theory.

Theorem ([Kawarabayashi & X 20])
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Theorem ([Kawarabayashi & X 20])
H is r-tractable ***if and only if*** *the apex subgraph of H is c-tractable.*

The theorem holds for directed graphs for an appropriate definition of apex.

Consequence

A strange problem

Input: Graph G and vertices x, y, z.

Output: A minimum cardinality set of edges F such that

- $d_{G-F}(x, y), d_{G-F}(y, z) \geq 3$,
- $d_{G-F}(x, z) \geq 4$.

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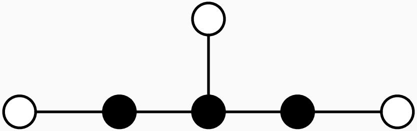

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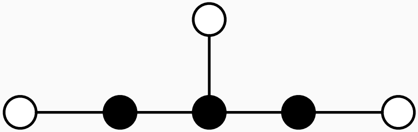

- $(3, 3)$-way-cut is NP-hard.
- $(4, 2)$-way-cut is NP-hard.

s-tractable graphs

Previously known

SHom(H)
Input: Graph G.

Output: Decide if there is a surjective homomorphism from G to H.

A graph H is s_0-tractable if **SHom**(H) is tractable.

- H is r-tractable then it is s-tractable.
- H is not s_0-tractable then it is not s-tractable.

[Elem, Hassin & Monnot 13]

Tractability from components

Theorem
A reflexive graph H is s-tractable if and only if each of its component is s-tractable.

Tractability from components

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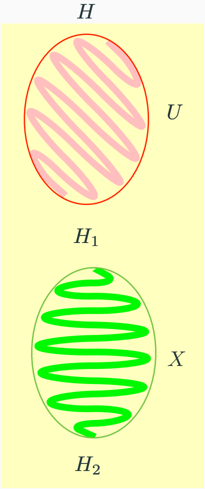
We will prove the case when H has two components.

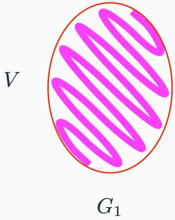
H is composed of components H_1 and H_2.

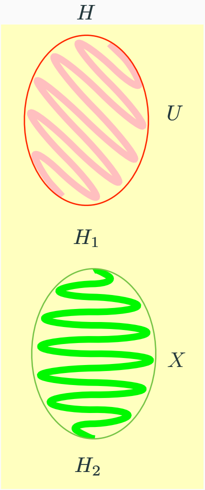
H_1 is not s-tractable. We reduce SVIO(H_1) to SVIO(H).

Hardness

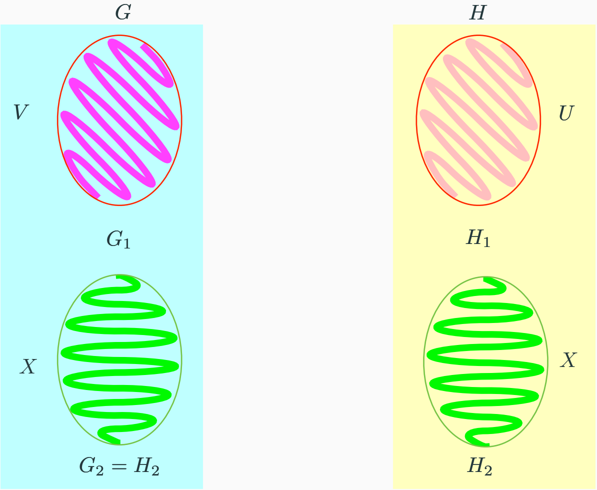


Hardness

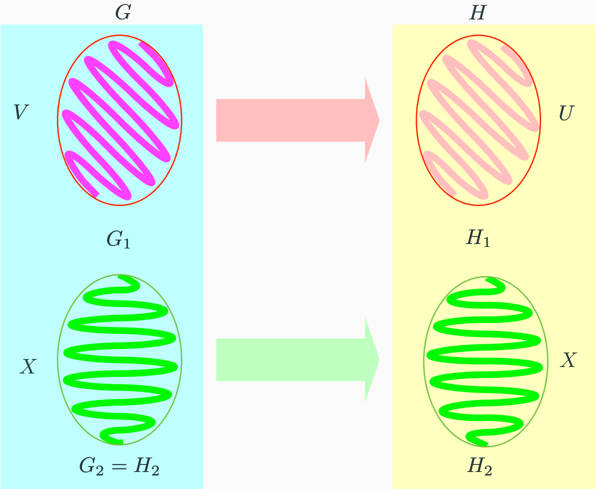




Hardness



Hardness



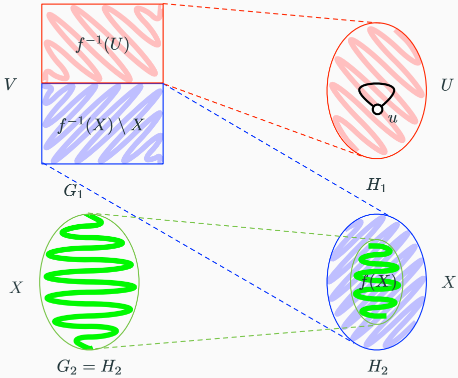
Hardness

Proof idea: For a minimum surjective map f from G to H, we can find a surjective map f' such that

- violation of f' is no larger than violation of f,
- $f'(X) = X$,
- $f'(V) = U$.

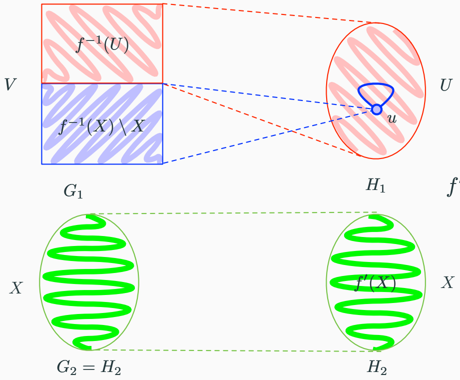
$f'|_V$ is the desired minimum violation map from G_1 to H_1.

Hardness



$$f(X) \subset X$$

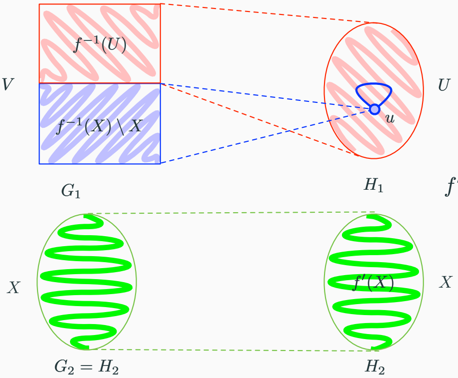
Hardness



$$f(X) \subset X$$

$$f'(v) = \begin{cases} v & \text{if } v \in X \\ u & \text{if } v \in f^{-1}(X) \setminus X \\ f(v) & \text{otherwise} \end{cases}$$

Hardness

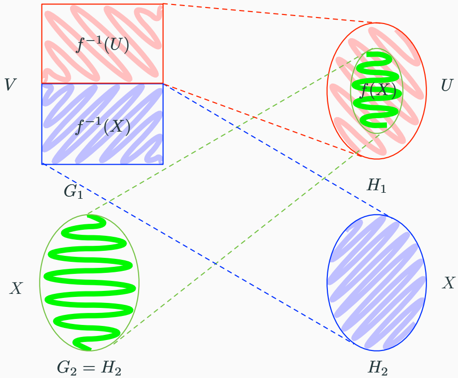


$$f(X) \subset X$$

$$f'(v) = \begin{cases} v & \text{if } v \in X \\ u & \text{if } v \in f^{-1}(X) \setminus X \\ f(v) & \text{otherwise} \end{cases}$$

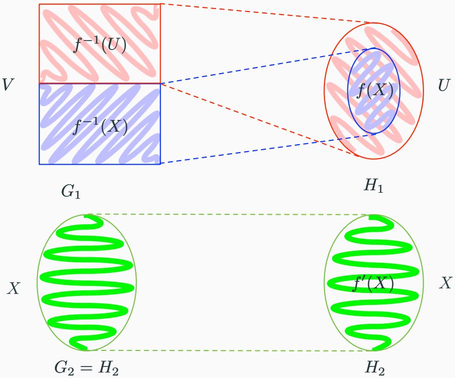
Reflexivity is crucial.

Hardness



$$f(X) \subset U$$

Hardness



$$f(X) \subset U$$

$$f'(v) = \begin{cases} v & \text{if } v \in X \\ f(f(v)) & \text{if } v \in f^{-1}(X) \\ f(v) & \text{otherwise} \end{cases}$$

The other direction: Polynomial time algorithm

Theorem
If the components of a reflexive graph H are s-tractable, then H is s-tractable.

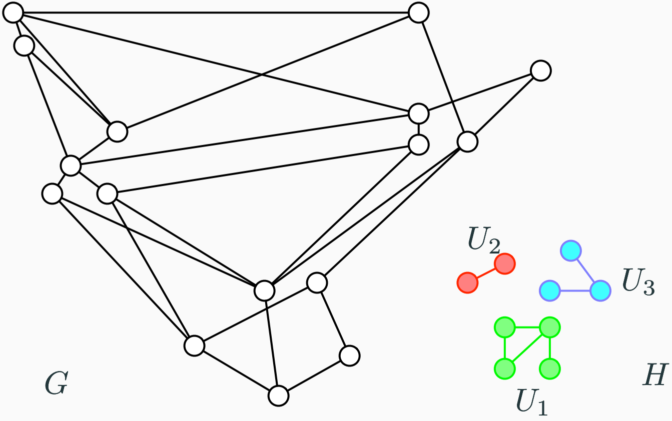
Set up

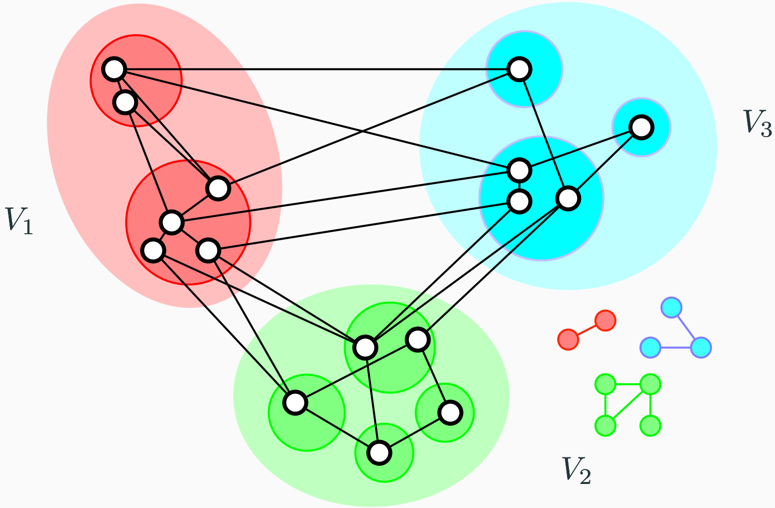
H is a k vertex graph consist of components $U_1, \ldots, U_\ell$.

$H[U_i]$ is s-tractable for all i.

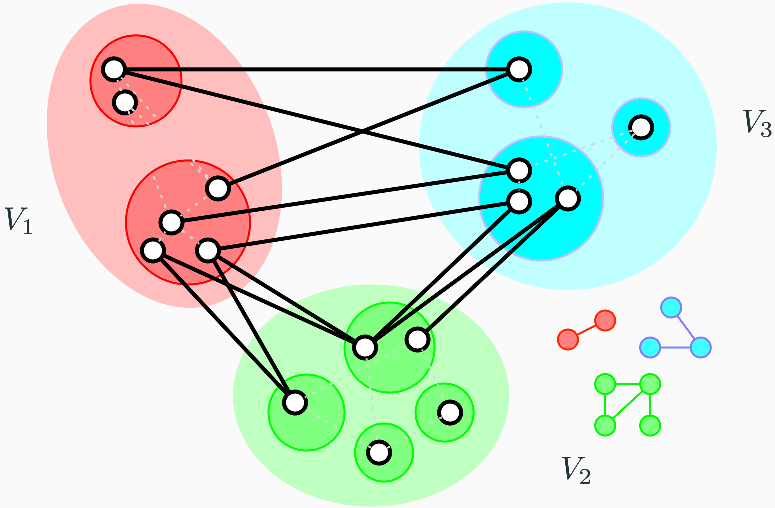
f is the optimal solution of SVIO(H) with input graph G.

$V_i = f^{-1}(U_i)$.


U_2
U_3
U_1
G
H



V_1
V_2
V_3



V_1
V_3
V_2

The set of edges crossing the ℓ-cut $(V_1, \ldots, V_\ell)$ has value at most the value of the min-k-cut of G.

min-k-cut value $\geq$ min violation $\geq$ ℓ-cut value.

Enumerating ℓ-cuts with small value

Theorem
There exists an algorithm that takes $n^{O(k)}$ time and produce all ℓ-cuts with value at most the value of a min-k-cut.

Enumerating ℓ-cuts with small value

Theorem
There exists an algorithm that takes $n^{O(k)}$ time and produce all ℓ-cuts with value at most the value of a min-k-cut.

Through spanning tree packing [Thorup 08; Chekuri, Quanrud, X 20].

Algorithm

Input graph G.

1. For each ℓ-partition $(V_1, \dots, V_\ell)$ of value at most min-k-cut
    1.1 Solve SVIO($H[U_i]$) with input $G[V_i]$.
    1.2 Combine the solutions into a candidate solution.
2. Output the minimum candidate solution.

Main Theorem

Theorem
A reflexive graph H is s-tractable if and only if each of its component is s-tractable.

Theorem
A reflexive tree T is s-tractable ***if and only if*** $\operatorname{diam}(T) \leq 4$ *and for every* 3 *distinct vertices at least 2 has distance at most* 3*.*

Trees

Theorem
A reflexive tree T is s-tractable ***if and only if*** $\text{diam}(T) \leq 4$ *and for every* 3 *distinct vertices at least 2 has distance at most* 3.

Proof idea: Hardess

- For trees the distance relations completely determines the existance of homomorphism.
- We design gadgets to force distance relations of fixed vertices.
- Conclude hardness from $(3, 3)$-way-cut and $(4, 2)$-way-cut.

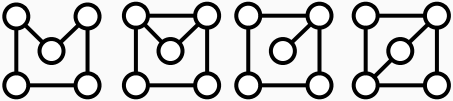
For the trees not covered by hardness, we see that T is r-tractable, therefore T is s-tractable.

Open Problems

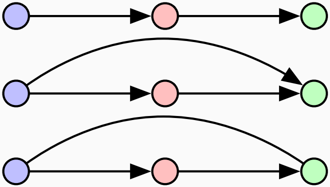
Classify the s-tractable graphs

All 4 vertex reflexive graphs and 2 vertex digraphs have been characterized.

- Conjectures
  - H has a surjective homomorphism to H', and H' is not s-tractable, then H is not s-tractable.
- 5 vertex graphs?



- 3 vertex digraphs?



Thank you