A Faster Pseudopolynomial Time Algorithm for Subset Sum

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The Subset Sum Problem

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Textbook DP algorithm due to Bellman that runs in $O(nt)$ **pseudopolynomial** time.

[Bellman '56]

Why pseudopolynomial time algorithm?

Faster pseudopolynomial time algorithm for subset sum implies faster polynomial time algorithms for various problems.

Applications

As a subroutine:

- knapsack
- scheduling
- graph problems with cardinality constraints

In practice:

- power indices (Voting Theory)
- set-based queries (Database)
- Subset sum based keys (Security)

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- First poly space algorithm: $\widetilde{O}(n^3 t)$ — **[Lokshtanov et al. '10]**

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Conditional lower bound: Subset sum solvable in $O(poly(n)t^{1-\epsilon})$ for any $\epsilon > 0$ implies faster algorithms for a wide variety of problems including set cover. [Bringmann '17]

Variants: Addition in $\mathbb{Z}_m$

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Different from the algorithm in $\mathbb{N}$!

Variants: multiset

Input: $2n$ natural numbers $x_1, x_2, x_3, \ldots, x_n, b_1, \ldots, b_n$ and a target number t.

Output: Does there exist non-negative integers $c_1, \ldots, c_n$, such that $\sum_{i=1}^{n} c_i x_i = t$ and $c_i \leq b_i$?

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    - $\widetilde{O}(t)$ time. **[Bringmann '17]**

Variants: Subset sums with cardinality constraint

Input: A set S of n natural numbers $x_1, x_2, x_3, \ldots, x_n$, cardinality constraint k and target number t.

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- Solvable in $O(knt)$ time by modifying Bellman's DP.
- We can solve it in $\tilde{O}(nt)$ time.

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- Our algorithm handles it with polylog factor slow down.
- We can also count the number of solutions faster than the standard dynamic programming algorithm.

Outline of the talk

We present two algorithms:

- Solve subset sum in $\mathbb{N}$.
- Solve subset sum in $\mathbb{Z}_m$.

Subset sums in $\mathbb{N}$

All Subset Sums

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Given a set S of n natural numbers and an (upper bound) u, compute all the realizable sums up to u.

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Finding all subset sums of S up to u: compute $\mathbf{\Sigma}(S) \cap [u]$.

Divide and conquer

Fact. If P and Q form a partition of a set S, then $\mathbf{\Sigma}(P) \oplus \mathbf{\Sigma}(Q) = \mathbf{\Sigma}(S)$.

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Straightforward divide-and-conquer algorithm for the all subset sums problem:

- Partition the set S into two sets
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- Combine them together with $\oplus$.

Review of the Bellman's dynamic programming algorithm

Input: A set S of n natural numbers $x_1, x_2, x_3, \ldots, x_n$ and an upper bound u.

Algorithm:

- $T_0 \leftarrow \{0\}$.
- $T_i \leftarrow T_{i-1} \cup \{s + x_i | s \in T_{i-1}, s + x_i \leq u\}$.

$O(nu)$ time.

Alternative view

Input: A set S of n natural numbers $x_1, x_2, x_3, \ldots, x_n$ and an upper bound u.

Algorithm:

- return $[u] \cap \bigoplus_{i=1}^{n} \mathbf{\Sigma}(\{x_i\})$.

$\mathbf{\Sigma}(\{x\}) = \{0, x\}$.

Convolution algorithm

Theorem. Given $A, B \subseteq [u]$, $A \oplus B$ can be computed in $O(u \log u) = \tilde{O}(u)$ time.

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Theorem. Given $A, B \subseteq [u] \times [v]$, $A \oplus B$ can be computed in $O(uv \log uv) = \tilde{O}(uv)$ time.

Two algorithms for all subset sums

If $S \subseteq [x..x+\ell]$, then we will show that $\mathbf{\Sigma}(S) \cap [u]$ can be found in

- $O(n(x+\ell))$ time. (Algorithm 1)
- $O((u/x)^2\ell)$ time. (Algorithm 2)

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We balance the running time of both algorithms to get the desired result.

Algorithm 1

Algorithm 1: Proof and analysis

Lemma *Given a set S of n numbers in $[x..x + \ell]$, one can compute the set of all subset sums $\mathbf{\Sigma}(S)$ in $\tilde{O}(n(x + \ell))$ time.*

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- $$T(n) = 2T(n/2) + \tilde{O}(n(x+\ell))$$
- Solves to $T(n) = \tilde{O}(n(x+\ell))$

$\square$

Algorithm 2

Algorithm 2: Idea

Lemma. *Given a set* $S \subseteq [x..x+\ell]$ *of size* n*, computing the set* $\mathbf{\Sigma}(S) \cap [u]$ *takes* $\widetilde{O}\big((u/x)^2\ell\big)$ *time.*

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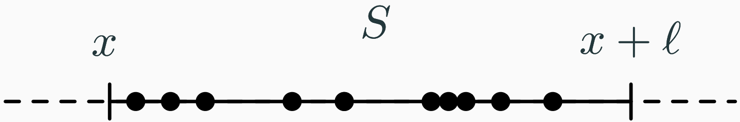
Proof Sketch. Same algorithm:

1. Partition S into L and R
2. Compute $\boldsymbol{\Sigma}(L) \cap [u]$ and $\boldsymbol{\Sigma}(R) \cap [u]$ recursively
3. Combine through (a smarter implementation of) $\oplus$.

Algorithm 2: A single recursive step



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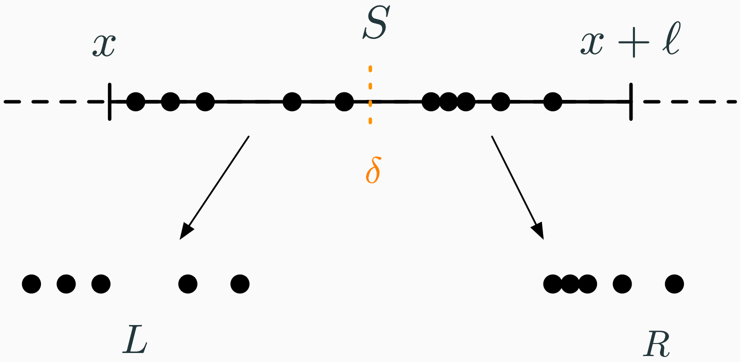

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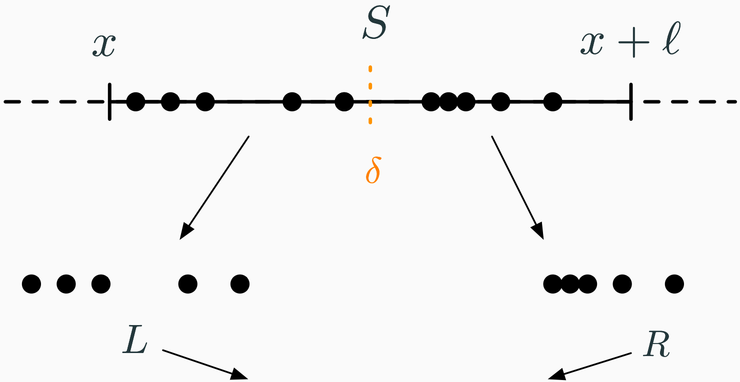

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- $z = ix + j$, where $i \in [k], j \in [\ell k]$.

Algorithm 2: A single recursive step

$$i \in [k], j \in [\ell k]$$

$$z = ix + j \qquad k = \left\lfloor \frac{u}{x} \right\rfloor$$

$$\Cap$$

$$\Sigma(L) \cap [u]$$

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Lift to 2D

$$i \in [k], j \in [\ell k] \quad \xrightarrow{\Phi} \quad (i, j)$$

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$\xrightarrow{\Phi}$

$A = \Phi(\Sigma(L) \cap [u])$

$B = \Phi(\Sigma(R) \cap [u])$

$A, B \subseteq [k] \times [\ell k]$

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$\Sigma(L) \oplus \Sigma(R)$ $\cap [u]$ $\xleftarrow{\Phi^{-1}}$ $A \oplus B$

$\tilde{O}(\ell k^2)$ = $\tilde{O}((u/x)^2 \ell)$ time

Algorithm 2: Run time analysis

Let $T(n, \ell)$ be the running time of Algorithm 2 with input set $S \subseteq [x..x + \ell]$ of size n.

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$\ell_1 + \ell_2 = \ell$.

$$
\begin{aligned}
T(n, \ell) &= T(n/2, \ell_1) + T(n/2, \ell_2) + \tilde{O}(\ell(u/x)^2) \\
&= \tilde{O}(\ell(u/x)^2)
\end{aligned}
$$

Algorithm 3

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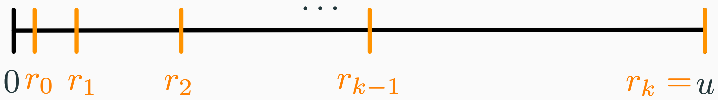
Algorithm

AllSubsetSum3(S, u):

- Partition $[u]$ into intervals $I_i = [r_{i-1}..r_i - 1]$ for $0 \leq i \leq k$.
- Let $S_i \leftarrow I_i \cap S$.
- Compute $\mathbf{\Sigma}(S_0)$ using Algorithm 1.
- Compute $\mathbf{\Sigma}(S_i)$ using Algorithm 2 for $1 \leq i \leq k$.
- Return $\bigoplus_{i=0}^{k} \mathbf{\Sigma}(S_i)$.

Algorithm 3



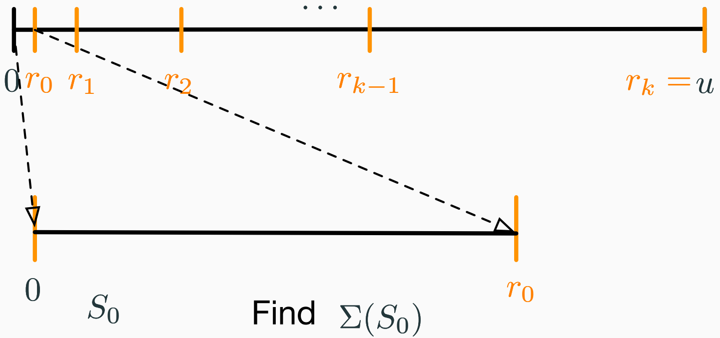
$$r_i = \lfloor 2^i r_0 \rfloor$$

$$S_i = S \cap [r_{i-1}..r_i - 1]$$

$$k = O(\log u)$$

$$n_i = |S_i|$$

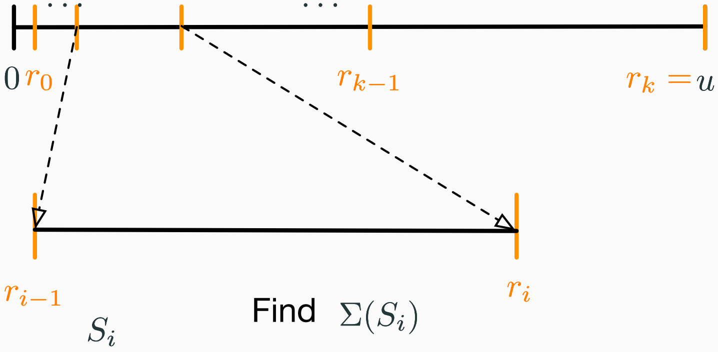
Algorithm 3



Find $\Sigma(S_0)$

Algorithm 1 $\tilde{O}(n_0 r_0)$

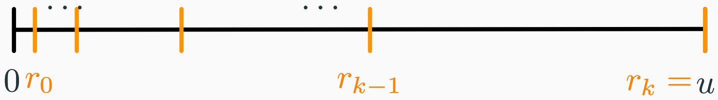
Algorithm 3



Find $\Sigma(S_i)$

Algorithm 2

$$\tilde{O}((\frac{u}{r_{i-1}})^2(r_i - r_{i-1})) = \tilde{O}(u^2/r_{i-1})$$



Find $\Sigma(S_i)$ for all $1 \leq i \leq k$

$$\sum_{i=1}^{k} \tilde{O}(\frac{u^2}{r_{i-1}}) = \tilde{O}(\frac{u^2}{r_0})$$

Algorithm 3: Analysis

- Find $\mathbf{\Sigma}(S_0)$ in $\tilde{O}(n_0 r_0) = \tilde{O}(\min(n, r_0) r_0)$ time.

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- Set $r_0 = u/\sqrt{n}$, we get $\tilde{O}(\sqrt{n} u)$.
- Set $r_0 = u^{2/3}$, we get $\tilde{O}(u^{4/3})$.

Lower bound?

There exist inputs $x_1 < \ldots < x_n$, such that any divide-and-conquer algorithm that computes $\mathbf{\Sigma}(S)$ by

- add parenthesis to this expression

$$\mathbf{\Sigma}(x_1) \oplus \ldots \oplus \mathbf{\Sigma}(x_n),$$

- compute all the intermediate output,

takes $\Omega(\min(\sqrt{n}t, t^{4/3}))$ time.

Subset sums in $\mathbb{Z}_m$

Overview of the result

$\mathbb{Z}_m = \{0, \ldots, m-1\}$, the integers modulo m.

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Let $S \subseteq \mathbb{Z}_m$ be a set of size n. $\boldsymbol{\Sigma}(S)$ can be found in $\widetilde{O}(\min(\sqrt{n}m, m^{5/4}))$ time.

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Basic number theory definition/facts

$\mathbb{Z}_m^* = \{x | x \in \mathbb{Z}_m, \gcd(x, m) = 1\}$, the set of units of $\mathbb{Z}_m$.

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The algorithm consists of a black box for solving subset sums when $S \subseteq \mathbb{Z}_m^*$, and then apply divide and conquer depending on the divisibility of the elements in S.

Subset sums in $\mathbb{Z}_m$

$S \subseteq \mathbb{Z}_m^*$

Segments

A segment of length ℓ is a set of the form $x[\ell] = \{0, x, 2x, \ldots, \ell x\}$. We denote $X[\ell] = \{ix | x \in X, i \in [\ell]\}$.

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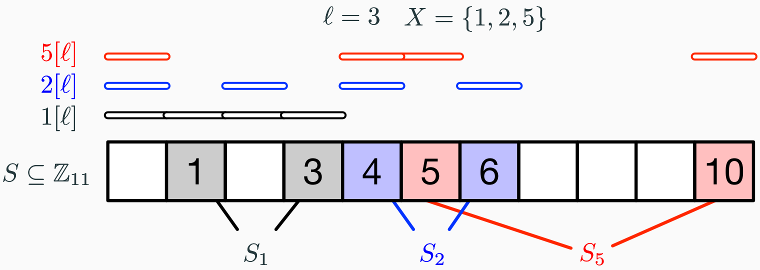
$\mathbf{\Sigma}(S)$ can be found quickly if S is covered by a segment.

Theorem

$S \subseteq \mathbb{Z}_m$ is a n element subset of $x[\ell]$, then $\mathbf{\Sigma}(S)$ can be found in $\tilde{O}(n\ell)$ time.

The algorithm when input is in $\mathbb{Z}_m^*$



We partition the input by segments.

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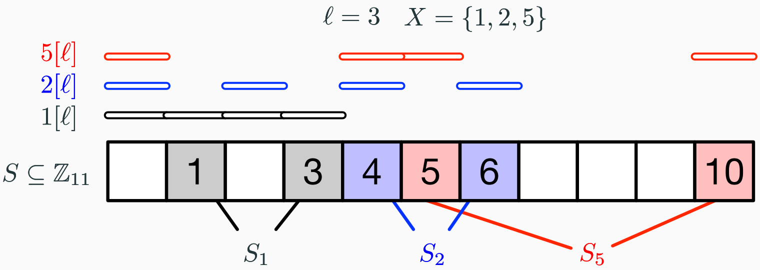

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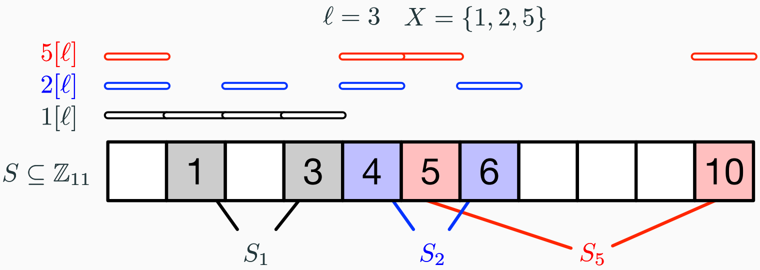

We partition the input by segments.

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- return $\bigoplus_{x \in X} \mathbf{\Sigma}(S_x)$.

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The total running time is $\tilde{O}(T(n, \ell, m) + n\ell + |X|m)$.

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The total running time is $\tilde{O}(T(n, \ell, m) + n\ell + |X|m)$. We need to find a small X that induces a cover of S, and we have to find one fast.

Covering $S \subseteq \mathbb{Z}_m^*$ by segments

Theorem

For any $S \subseteq \mathbb{Z}_m^*$, there exists a $x \in \mathbb{Z}_m^*$, such that $|S \cap x[\ell]| = \Omega(\frac{\ell}{m}|S|)$.

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- For any subset $S \subseteq \mathbb{Z}_m^*$, there is a $x[\ell]$ that covers $|S|\frac{\ell}{m}$ elements in S in expectation.

Cover S with segments

Algorithm

$\textsc{GreedySetCover}(S \subseteq \mathbb{Z}_m^*)$

1. Pick $x[\ell]$ such that $|x[\ell] \cap S|$ is maximized.
2. $S \leftarrow S \setminus x[\ell]$
3. $\textsc{GreedySetCover}(S)$

Finds a cover of size $O(\frac{m}{\ell} \log n)$ in $O(n\ell)$ time.

Subset sums in $\mathbb{Z}_m^*$

Theorem

All subset sums with input $S \subseteq \mathbb{Z}_m^*$ can be solved in $\tilde{O}(\sqrt{n}m)$ time.

Proof.

$$\tilde{O}(T(n,\ell,m) + n\ell + (\frac{m}{\ell})m) = \tilde{O}(\frac{m^2}{\ell} + n\ell)$$

Let $\ell = \frac{m}{\sqrt{n}}$. □

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Theorem ([Hamidoune, Llad & Serra 08])

If $S \subseteq \mathbb{Z}_m^*$ and $|S| \geq 2\sqrt{m}$, then $\boldsymbol{\Sigma}(S) = \mathbb{Z}_m$.

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Definitions

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We define $\textsc{AllSubsetSums}(S, m, d)$ as an algorithm that finds all subset sums of S in $\mathbb{Z}_m$, if $S \subseteq \mathbb{Z}_{m,d}$

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We solved the case for $\textsc{AllSubsetSums}(S, m, 1)$.

$$\mathbf{\Sigma}(S) = \textsc{AllSubsetSums}(S, m, m)$$

The algorithm for all subset sums in $\mathbb{Z}_m$

- $S/p = \{s/p : s \in S, p | s\}$
- $S\%p = \{s : s \in S, p \not| s\}$

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Algorithm

$\textsc{AllSubsetSums}(S, m, d)$:

1. $d = 1$, use the previous algorithm.
2. $p \leftarrow$ the largest prime factor of d
3. **[All elements in S divisible by p]**
   $A \leftarrow \textsc{AllSubsetSums}(S/p, m/p, d/p)$
4. **[All elements in S not divisible by p]**
   $B \leftarrow \textsc{AllSubsetSums}(S\%p, m, d/p)$
5. return $(p \cdot A) \oplus B$

Example recursion tree where $S = \mathbb{Z}_6$

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0	1	2	3	4	5

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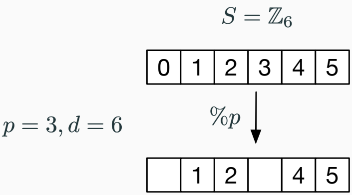

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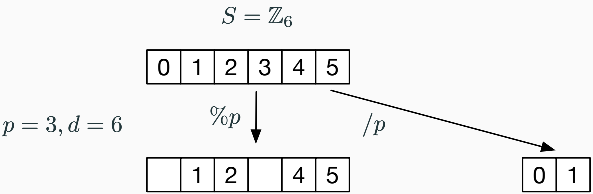

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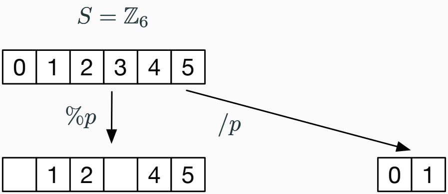

$p = 3, d = 6$

$p = 2, d = 2$

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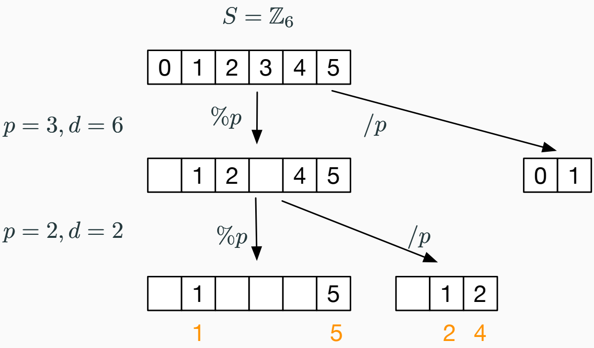

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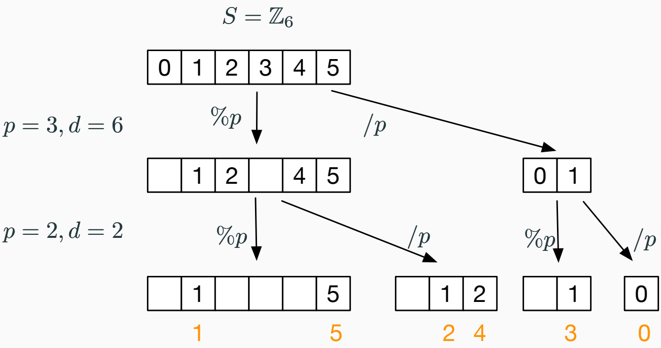

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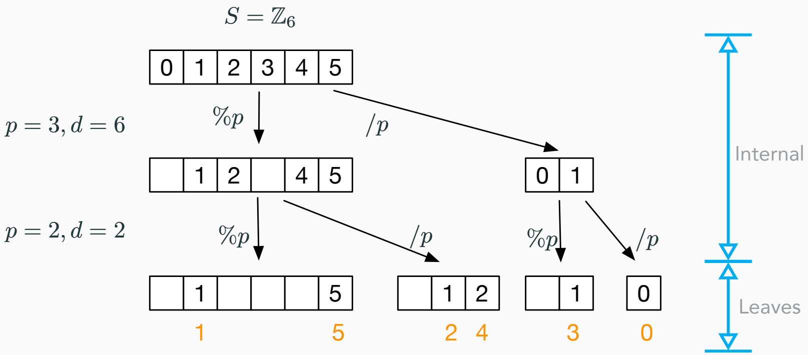

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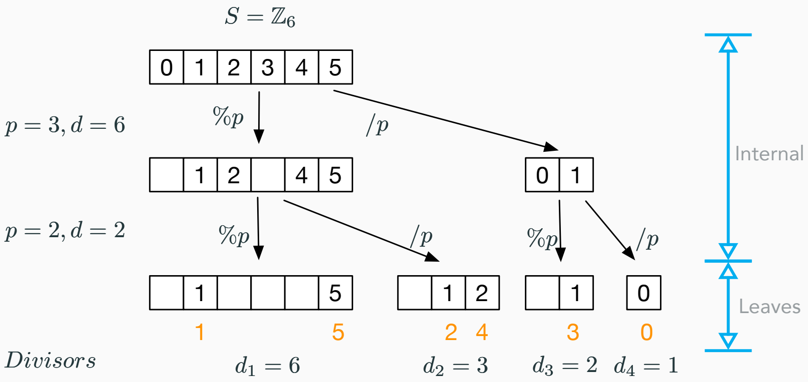

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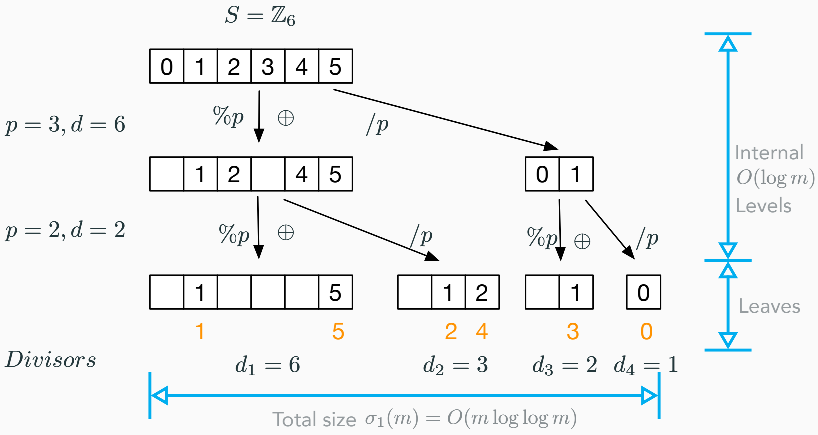
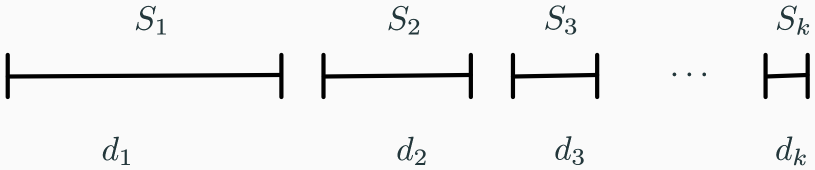


$\sigma_i(m) = \sum_{d|m} d^i$.

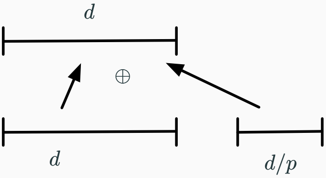
Run time analysis: Leaves



Compute $\mathbf{\Sigma}(S_i)$ for each i. $|S_i| = n_i$. $d_i \leq m/i$ is the ith largest divisor of m.

$$\begin{aligned}
&\tilde{O}(\sum_i \min(\sqrt{n_i} d_i, d_i^{5/4})) \\
=&\tilde{O}(\sum_i \min(\sqrt{n_i} m/i, (m/i)^{5/4})) \\
=&\tilde{O}(\min(\sqrt{n} m, m^{5/4}))
\end{aligned}$$

Run time analysis: Internal nodes



- There are $O(\log m)$ levels.
- Each level, the time spent on $\oplus$ is $\tilde{O}(\sum_{d|m} d) = \tilde{O}(\sigma_1(m)) = \tilde{O}(m)$.
- The total running time over internal nodes are $\tilde{O}(m)$.

Run time analysis

Theorem

All subset sums in $\mathbb{Z}_m$ can be solved in $\tilde{O}(\min(\sqrt{n}m, m^{5/4}))$.

Open Problems

Open Problems: Deterministic near linear time algorithm

Is there a deterministic $\widetilde{O}(t)$ time algorithm for the subset sum problem matching its conditional lower bound?

Open Problems: Output sensitive subset sum

Let $k = |\mathbf{\Sigma}(S) \cap [t]|$. Assume $k \ll t$.

- Known: subset sum in $O(nk)$ time use Bellman's DP algorithm.
- Can we obtain an algorithm with $\widetilde{O}(\sqrt{n}k)$ running time?

Open Problems: Covering $\mathbb{Z}_m$ by segments of length ℓ

Let $f(m, \ell)$ be the minimum number of segments of length ℓ required to cover $\mathbb{Z}_m$.

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Theorem ([Chen, Shparlinski & Winterhof '13])

- $f(m, \ell) = O(\frac{m}{\ell})$ *if m is prime.*
- $f(m, \ell) = \frac{m^{1+o(1)}}{\sqrt{\ell}}$.

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Theorem ([Koiliaris & Xu '17])

$f(m,\ell) = \sigma_0(m) + O(\sigma_1(m) \log m/\ell) = \frac{m^{1+o(1)}}{\ell}$

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Conjecture: $f(m,\ell) = O(\frac{m}{\ell})$

Thank you