Detecting Weakly Simply Polygon

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Motivation

- Many algorithms that works for simple polygons could work for degenerate cases.
- We define weakly simple polygon, which intuitively means the polygon overlaps but doesn’t cross itself.
- Moreover, we provide an efficient algorithm to decide if a polygon is weakly simple.

Definition

- A closed curve P on the plane is weakly simple if for any $\varepsilon > 0$, there is a simple closed curve P' whose Fréchet distance from P is at most ε.

A useful Characterization

Ribó Mor’s recent result implies a polygon is weakly simple iff for every $\varepsilon > 0$ we can obtain a simple polygon by perturbing each vertex within a ball of radius ε. [Ribó Mor 2006]

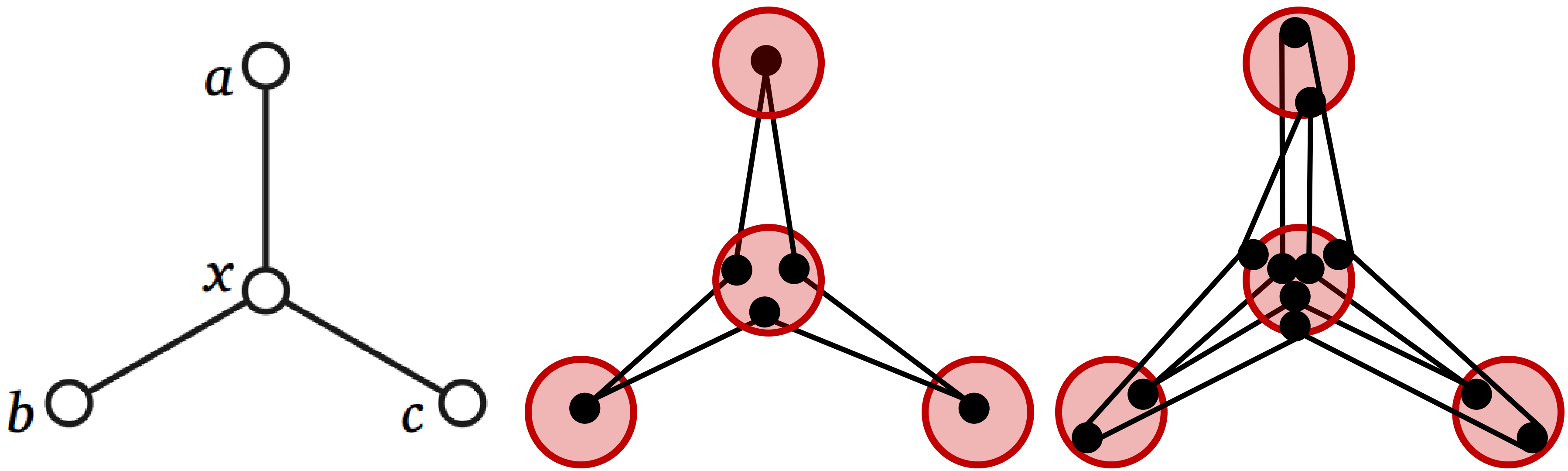
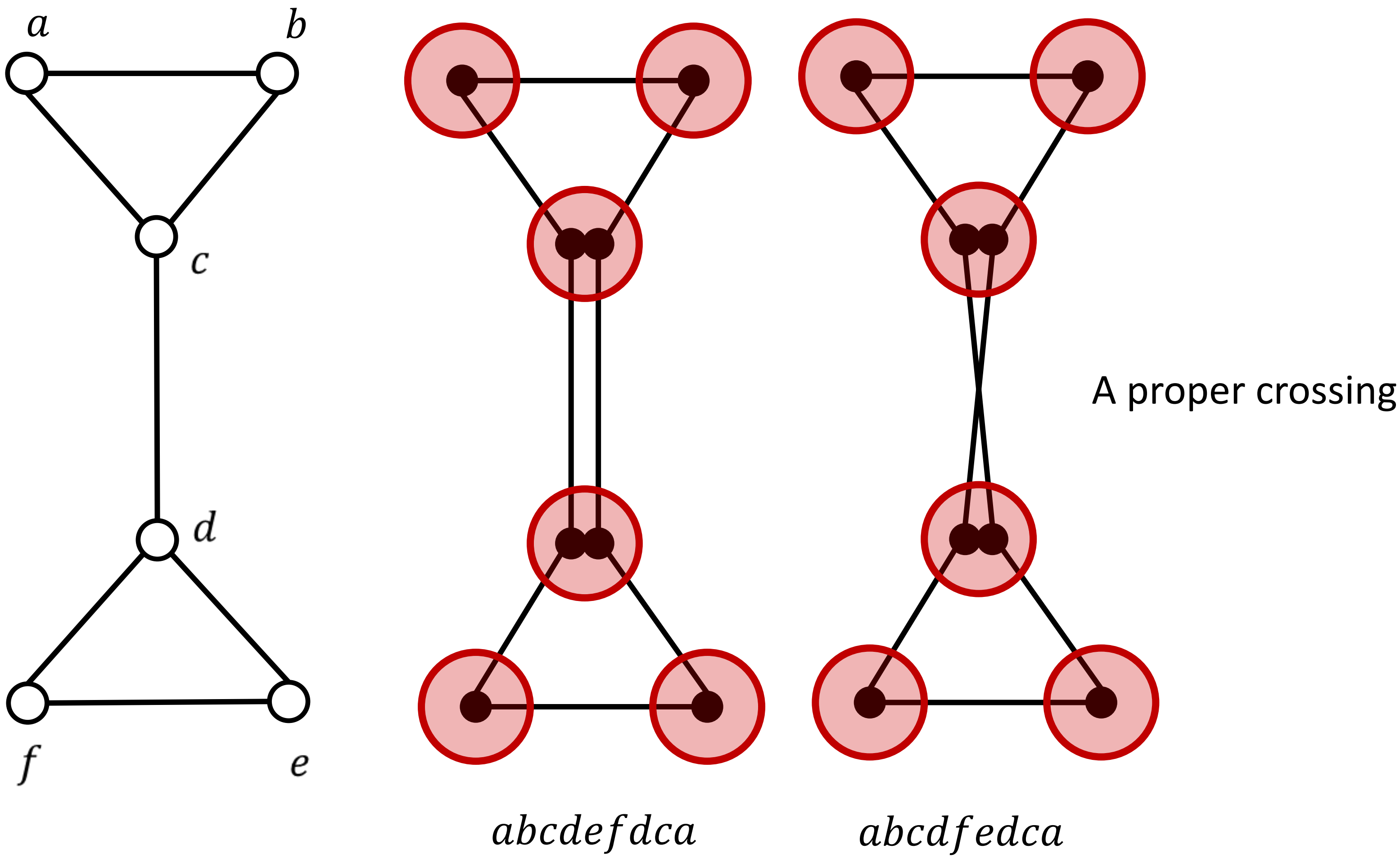


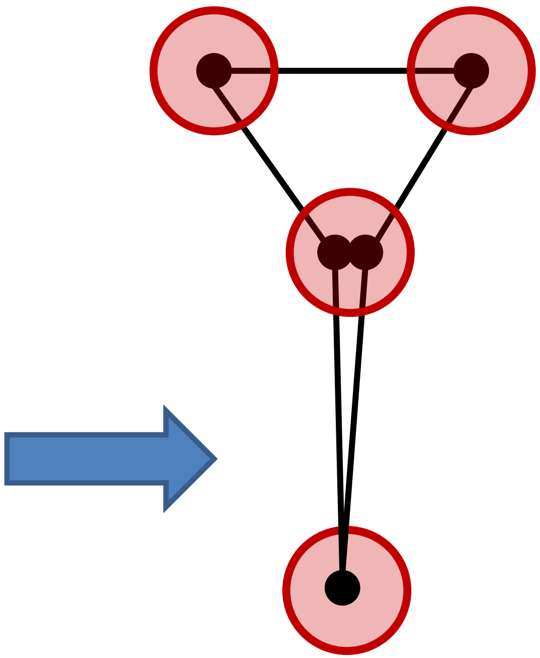
Figure 1. $axbxcxa$ is weakly simple; $axbxcxaxbxcxa$ is not.

There are alternative definitions which fails to capture the weakly simple property for various reasons. For example, Toussaint defined weakly simple as polygons with turning number ± 1 and no subpath have a "proper crossing". [Toussaint 1989]



The right side is a "zoomed in view". Each large disk is one node, and the line segments between the same large disks overlaps.

Spurs: vertices whose two incident edges overlap.

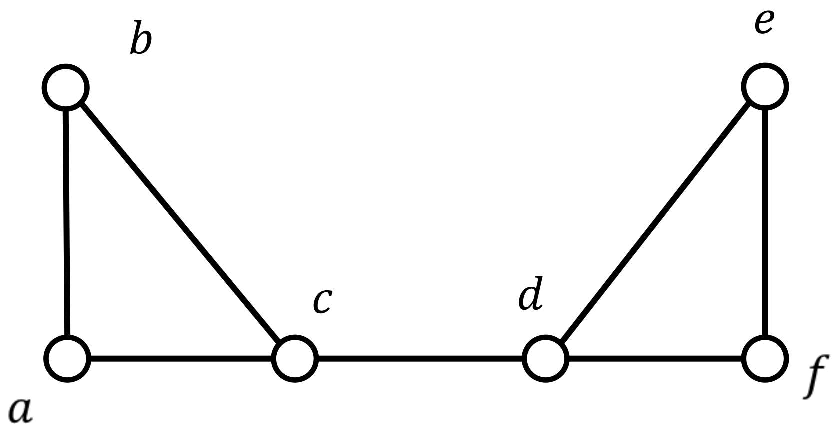


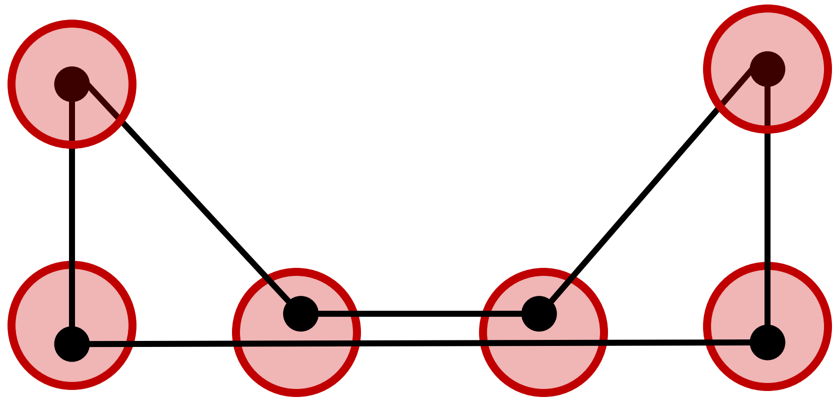
A spur

Toussaint's definition fails because turning numbers are not well defined.

Forks: A vertex is a fork if it is contained in the interior of some edges.

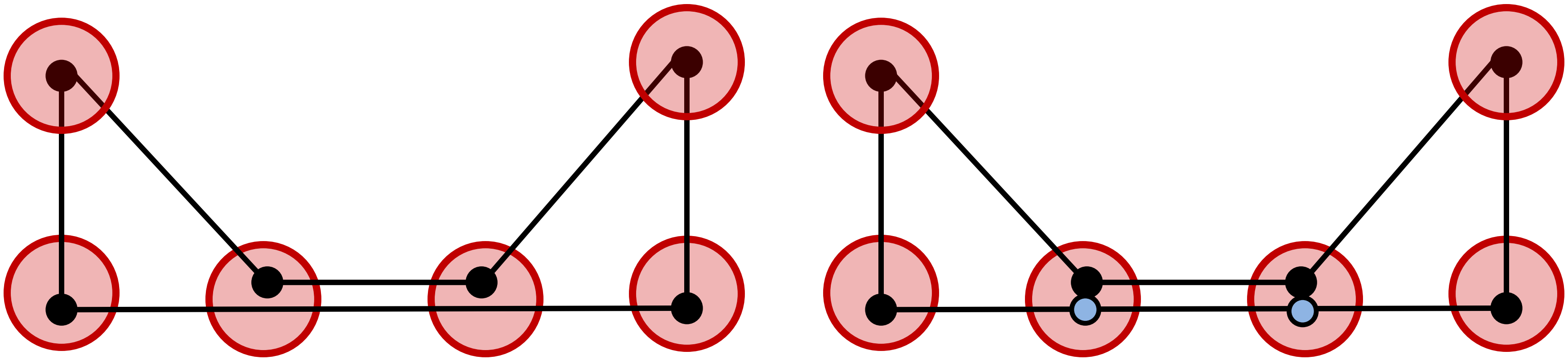
$abcdefa$





Forks: c and d are contained inside the edge af.

Polygon with forks can be converted to equivalent polygon without forks with $O(n^2)$ blow up.



$O\big(f(n)\big)$ algorithm (without forks) $\Rightarrow O\big(f(n^2)\big)$ algorithm (with forks)

Detecting weakly simple polygon without forks

With spurs?	Time (n = number of vertices)	Reference
No Spurs	$O(n)$	Folklore
With spurs	$O(n^3)$	[Cortese et al. 2009]
With spurs	$O(n \log n)$	This paper

Instead of a polygon without forks, we can consider the polygon to be a walk on a plane graph. Similarly we can define a weakly simple walk of a plane graph.

We want to decide if a walk W on the plane graph G is weakly simple.

An edge uv is a base for v if each occurrence of v in the walk W is immediately preceded or followed by the other endpoint u.

An edge uv is expandable if

1. uv is a base for both u and v.
2. uv is the unique base either u or v.

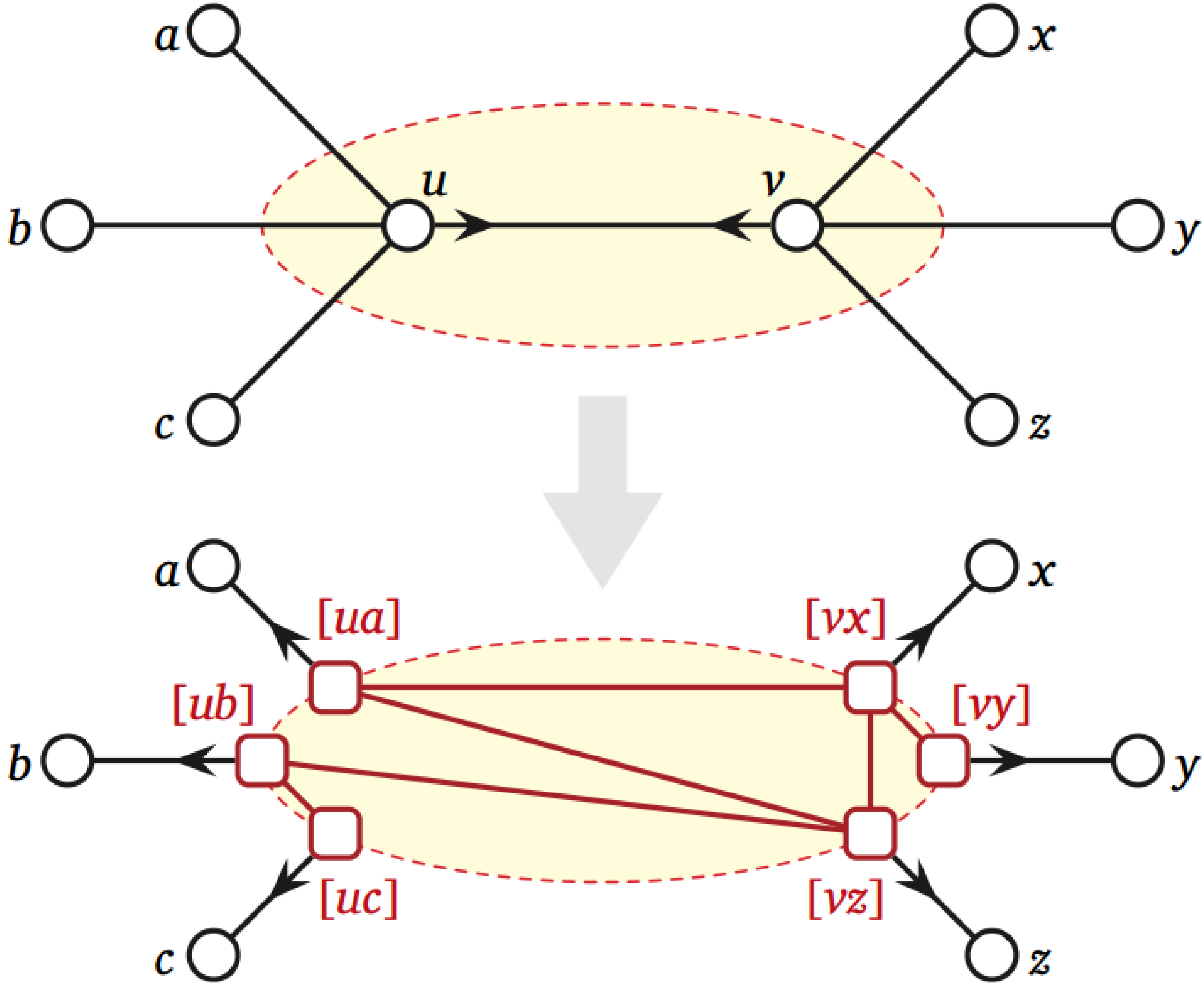
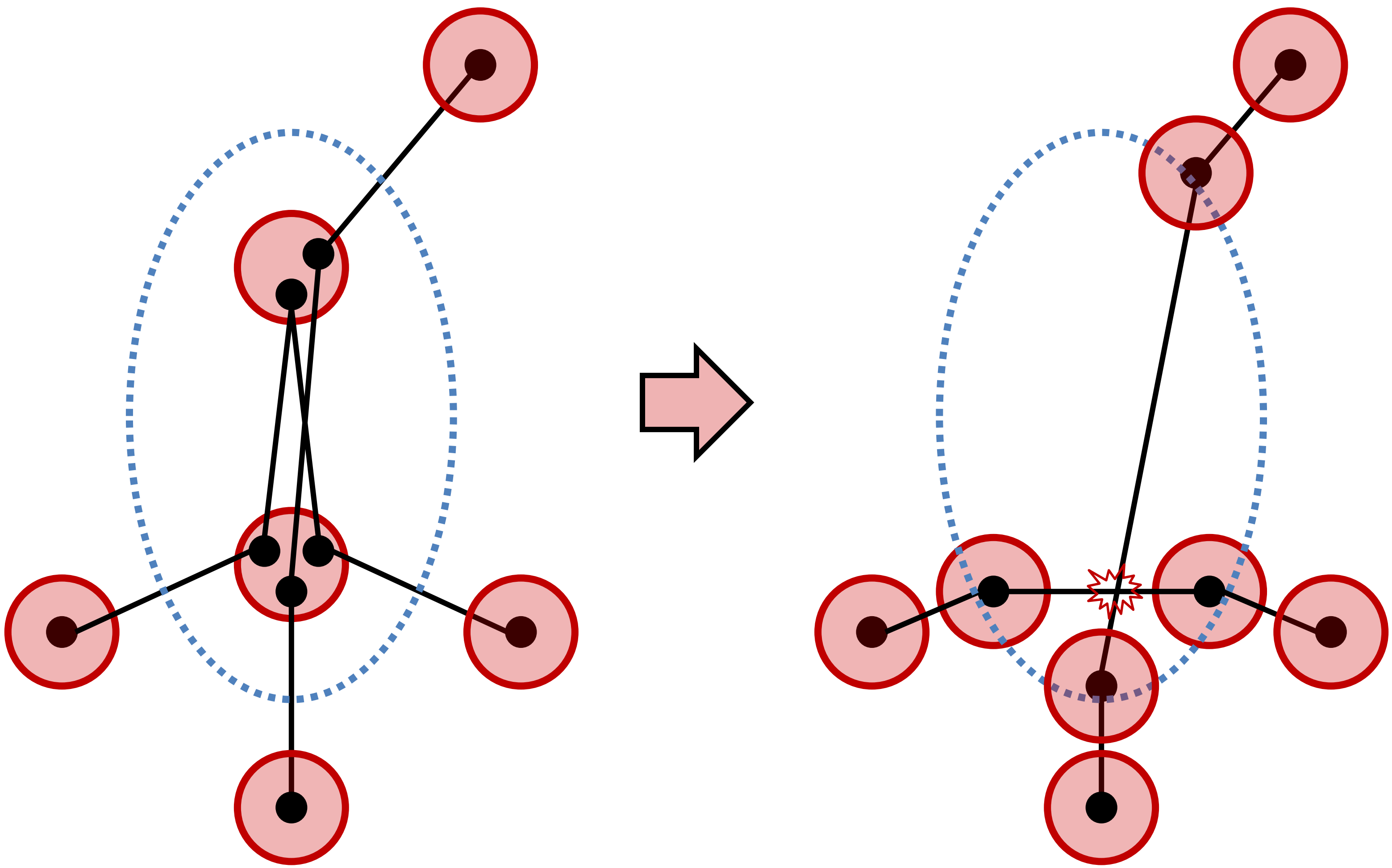


Figure 2. An example of edge expansion. An arrow leaving a node indicates the base of that node.

The walk is weakly simple iff the walk after the expansion is weakly simple.
The expansion might introduces local non-planarity.



The algorithm:

1. A linear time preprocessing stage, after preprocessing every vertex has an base.
2. Pick an expandable edge, expand it. Repeat this step until there is no more expandable edge. Reject if any expansion introduce local non-planarity.
3. The graph is either a cycle or a empty graph, and it’s easy to check if it’s weakly simple.

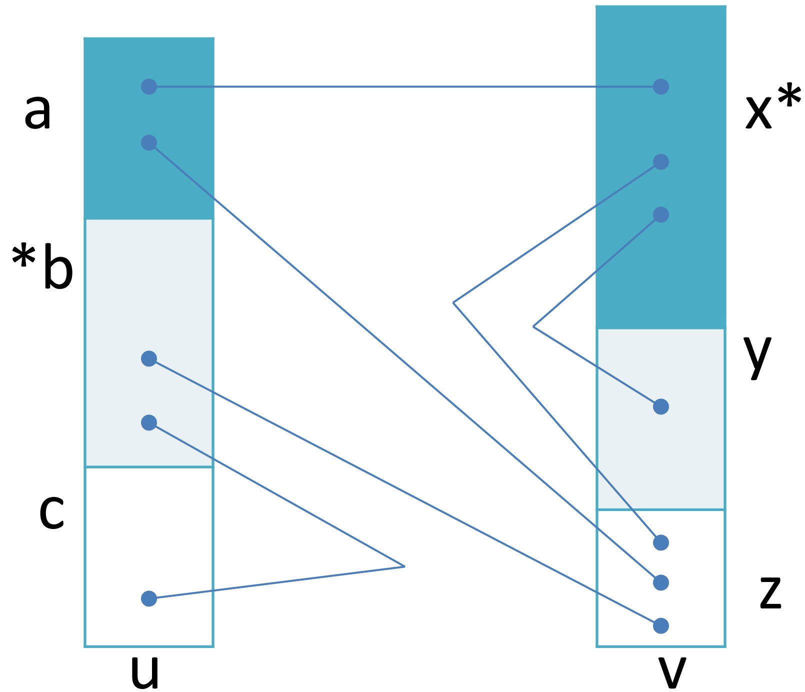
Analysis

Define a potential function $\Phi(W, G) = |\mathrm{W}| - |\mathrm{G}|$. Notice $0 \leq \Phi(W, G) \leq n$, and every expansion decrease $\Phi(W, G)$ by at least 1. There are at most n expansions.

The naïve implementation represent the walk by a circular string. The the expansion modify the walk locally by replacing strings of the form $a(uv)^k x$ to $a[ua][vx]x$.

Each expansion can take $O(n)$ time, thus the algorithm takes $O(n^2)$ time.

We can reuse the nodes instead of delete them. Namely we let $[ua] = u$ and $[vx] = v$ for some a and x.


a
x*
*b
y
c
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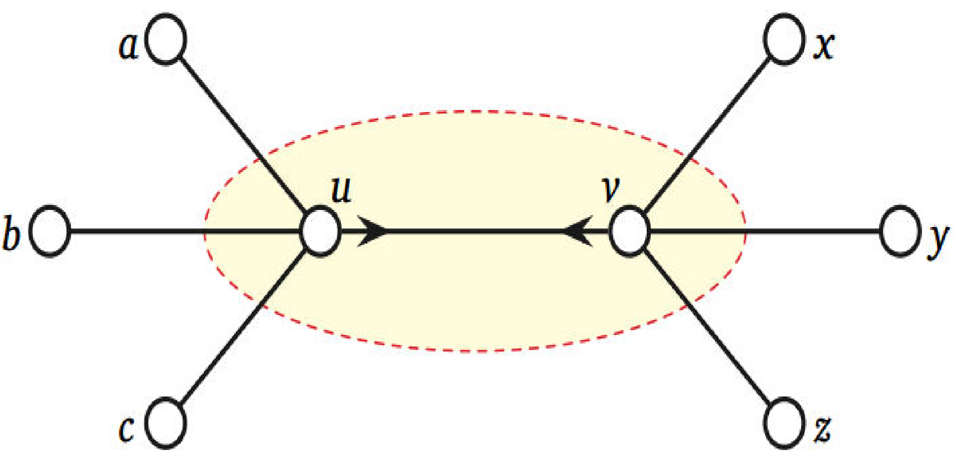
u
v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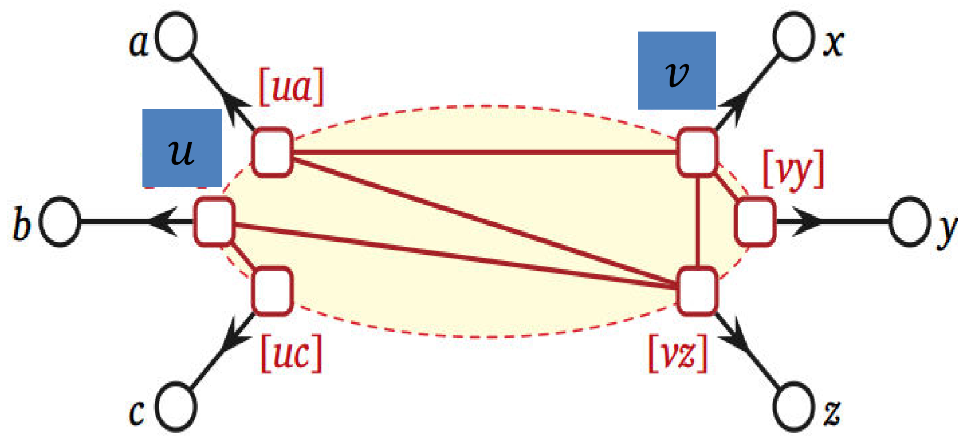
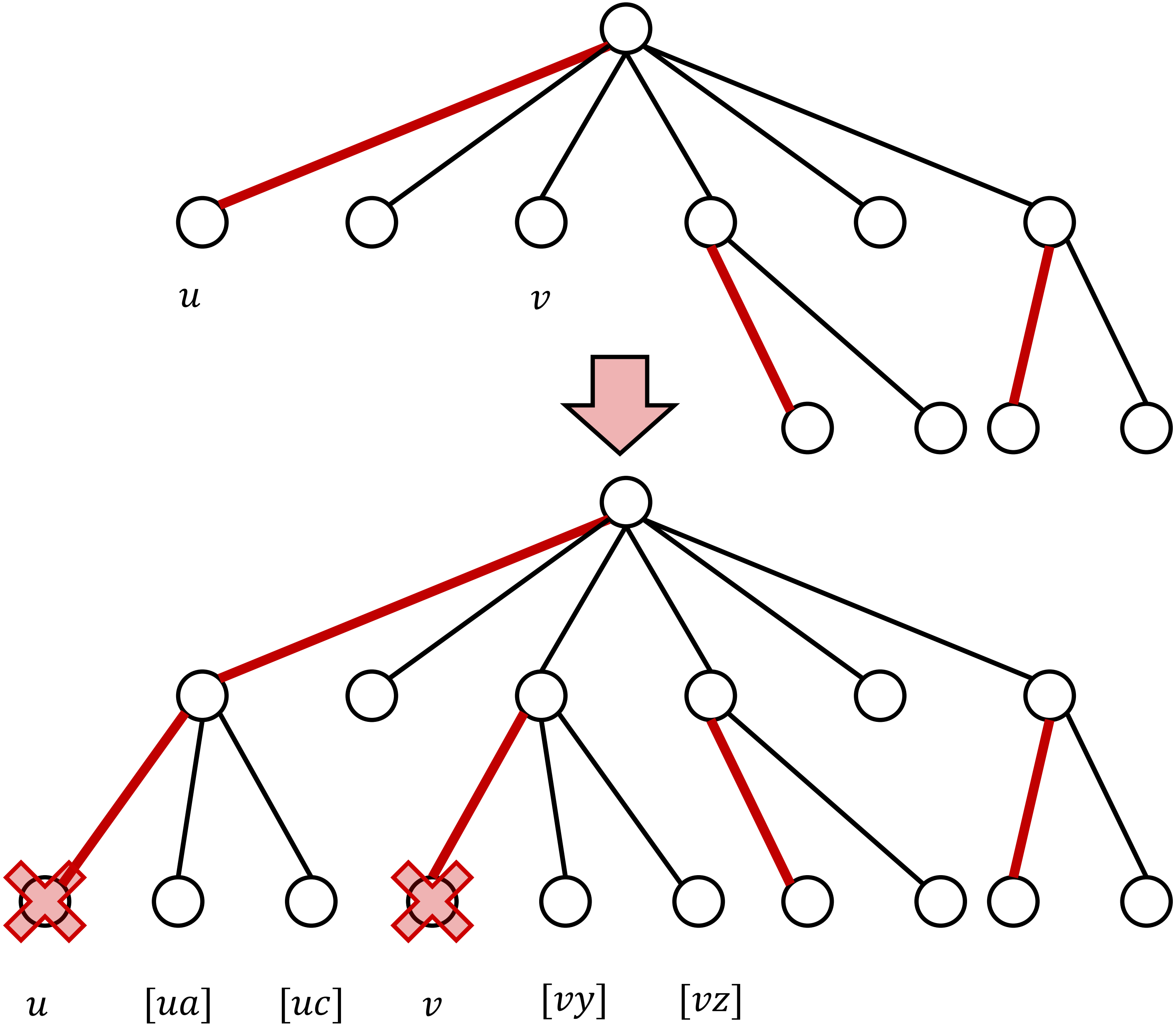

a
x
[ua]
v
u
[vy]
b
y
[uc]
[vz]
c
z

- $w(u)$ and $w'(u)$ measure the number of times the walk crosses the vertex u before and after the expansion.
- The time spent on expanding an edge uv equals $O\big(w(u) + w(v)\big)$ if we do not reuse the vertices.
- It will run in $O((w(u) - w'(u)) + (w(v) - w'(v)))$ time if we reuse the vertices.
- Heavy-light decomposition implies the running time to be $O(n \log n)$.


u
v
u
[ua]
[uc]
v
[vy]
[vz]

References

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